GENERALISATION OF KELVIN'S FUNCTIONS

By

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ABSTRACT: From the definitions of the three kinds of Bessel functions of two types of several arguments and unrestricted order KELVIN'S ber, bei, ker, kei, her and hei functions of two types of several arguments and unrestricted order have been defined; some formulae for these functions have also been derived.

1. DEFINITIONS OF THE FIRST KIND OF GENERALISED BESSEL FUNCTIONS

Taking $J_\mu^{(1)}\{x_n\}$, $J_\mu^{(2)}\{x_n\}$ the first and the second type of the Bessel functions of the first kind and of order μ and n arguments $x_1, \ldots, x_n$ defined by equations

$$J_\mu^{(1)}\{x_n\} = \sum^{\infty} J_{t_1}(x_1) \ldots J_{t_n}(x_n), \tag{1}$$

$$t_1, \ldots, t_n = -\infty \text{ except } t_r, \text{ which is given by}$$

$$t_1 + \ldots + nt_n = \mu$$

$$J_\mu^{(2)}\{x_n\} = \sum^{\infty} J_{t_1}(x_1)\; I_{t_2}(x_2)\; J_{t_3}(cx_3) \ldots \tag{2}$$

$$t_1, \ldots, t_n = -\infty \text{ except } t_r, \text{ which is given by}$$

$$t_1 + \ldots + nt_n = \mu$$

where the last term within the summation is $J_{t_n}(x_n)$ or $I_{t_n}(x_n)$ according as n is odd or even; the subscript r in t_r may have any integral value from 1 to n; similarly taking $I_\mu^{(1)}\{x_n\}$, $I_\mu^{(2)}\{x_n\}$ the first and the second type of the modified Bessel functions of the first kind of order μ and n arguments $x_1, \ldots, x_n$ defined by the equations

$$I_\mu^{(1)}\{x\}_n = \sum^{\infty} I_{t_1}(x_1) \ldots I_{t_n}(x_n) \tag{3}$$

$$t_1, \ldots, t_n = -\infty \text{ except } t_r, \text{ which is given by}$$

$$t_1 + \ldots + nt_n = \mu$$

and

$$I_\mu^{(2)}\{x_n\} = \sum^{\infty} I_{t_1}(x_1)\, J_{t_2}(x_2)\, I_{t_3}(x_3) \ldots \tag{4}$$

$$t_1, \ldots, t_n = -\infty \text{ except } t_r, \text{ which is given by}$$

$$t_1 + \ldots + nt_n = \mu$$

where the last term within the summation is $I_{t_n}(x_n)$ or $J_{tn}(x_n)$ according as n is odd or even,
we can generalise ber, bei, ker, kei, her and hei functions of one argument and order μ [2, p. 316].

For simplicity $x_1, \ldots, x_n$ and μ can be taken as real quantities, later on by suitable modifications $x_1, \ldots, x_n$ and μ may be given any general value.

2. The second and the third kind of generalised Bessel functions

The two types of the second kind of generalised Bessel functions $Y_\mu^{(1)}\{x_n\}$, $Y_\mu^{(2)}\{x_n\}$ and modified Bessel functions $K_\mu^{(1)}\{x_n\}$, $K_\mu^{(2)}\{x_n\}$ are defined by the equations

$$Y_\mu^{(1)}\{x_n\} = \frac{J_\mu^{(1)}\{x_n\}\cos\mu\pi - J_{-\mu}^{(1)}\{(-1)^{n+1}x_n\}}{\sin\mu\pi},$$

$$Y_\mu^{(2)}\{x_n\} = \frac{J_\mu^{(2)}\{x_n\}\cos\mu\pi - J_{-\mu}^{(2)}\{x_n\}}{\sin\mu\pi}; \tag{1}$$

$$K_\mu^{(1)}\{x_n\} = \frac{\pi}{2}\frac{I_{-\mu}^{(1)}\{x_n\} - I_\mu^{(1)}\{x_n\}}{\sin\mu\pi},$$

$$K_\mu^{(2)}\{x_n\} = \frac{\pi}{2}\frac{I_{-\mu}^{(2)}\{(-1)^{n+1}x_n\} - I_\mu^{(2)}\{x_n\}}{\sin\mu\pi}. \tag{2}$$

In case $\mu = 0, \pm 1, \pm 2, \ldots = m$ say, from § 1 (1 — 4) it is found that

$$J^{(1)}_{-m}\{x_n\} = (-1)^m J^{(1)}_m \{(-1)^{n+1} x_n\},\ J^{(2)}_{-m}\{x_n\} = (-1)^m J^{(2)}_m \{x_n\}; \quad (3)$$

$$I^{(1)}_{-m}\{x_n\} = I^{(1)}_m \{x_n\}, \qquad I^{(2)}_{-m}\{x_n\} = I^{(2)}_m \{(-1)^{n+1} x_n\}; \quad (4)$$

and, therefore, the numerators and the denominators both vanish in (1) and (2). So in this case the second kind of functions on the left-hand side of (1) and (2) are defined by the limit of the expressions on the right-hand side as $\mu \to m$.

The two types of the third kind of the generalised Bessel functions for each of the first and the second Hankel functions are defined by the equations

$$H^{(1),(1)}_\mu \{x_n\} = J^{(1)}_\mu \{x\} + i\, Y^{(1)}_\mu \{x_n\},\ H^{(2),(1)}_\mu \{x_n\} = J^{(1)}_\mu \{x\} - i\, Y^{(1)}_\mu \{x_0\}; \quad (5)$$

$$H^{(1),(2)}_\mu \{x_n\} = J^{(2)}_\mu \{x_n\} + i\, Y^{(2)}_\mu \{x_n\},\ H^{(2),(2)}_\mu \{x_n\} = J^{(2)}_\mu \{x_n\} - i\, Y^{(2)}_\mu \{x_n\}, \quad (6)$$

3. Generalisation of Kelvin's functions

Now, Kelvin's $\mathrm{ber}_\mu\ (x_1)$, $\mathrm{bei}_\mu\ (x_1)$, $\mathrm{ker}_\mu\ (x_1)$, $\mathrm{kei}_\mu\ (x_1)$, $\mathrm{her}_\mu\ (x_1)$ and hei (x_1) functions of order μ and one argument x_1 can be generalised to the functions of n arguments $x_1, \ldots, x_n$ by the equations [1, p. 81]

$$J^{(1)}_\mu \{x_n\, e^{\frac{3}{4}n\pi i}\} = \mathrm{ber}^{(1)}_\mu \{x_n\} + i\, \mathrm{bei}^{(1)}_\mu \{x_n\}, \quad (1)$$

$$J^{(2)}_\mu \{x_n\, e^{\frac{3}{4}n\pi i}\} = \mathrm{ber}^{(2)}_\mu \{x_n\} + i\, \mathrm{bei}^{(2)}_\mu \{x_n\}; \quad (2)$$

$$K^{(1)}_\mu \{x_n\, e^{\frac{1}{4}n\pi i}\} = \mathrm{ker}^{(1)}_\mu \{x_n\} + i\, \mathrm{kei}^{(1)}_\mu \{x_n\}, \quad (3)$$

$$K^{(2)}_\mu \{x_n\, e^{\frac{1}{4}n\pi i}\} = \mathrm{ker}^{(2)}_\mu \{x_n\} + i\, \mathrm{kei}^{(2)}_\mu \{x_n\}: \quad (4)$$

$$H^{(1),(1)}_\mu \{x_n\, e^{\frac{3}{4}n\pi i}\} = \mathrm{her}^{(1),(1)}_\mu \{x_n\} + i\, \mathrm{hei}^{(1),(1)}_\mu \{x_n\},$$
$$H^{(2),(1)}_\mu \{x_n\, e^{\frac{3}{4}n\pi i}\} = \mathrm{her}^{(2),(1)}_\mu \{x_n\} + i\, \mathrm{hei}^{(2),(1)}_\mu \{x_n\}; \quad (5)$$

$$H^{(1),(2)}_\mu \{x_n\, e^{\frac{3}{4}n\pi i}\} = \mathrm{her}^{(1),(2)}_\mu \{x_n\} + i\, \mathrm{hei}^{(1),(2)}_\mu \{x_n\},$$
$$H^{(2),(2)}_\mu \{x_n\, e^{\frac{3}{4}n\pi i}\} = \mathrm{her}^{(2),(2)}_\mu \{x_n\} + i\, \mathrm{hei}^{(2),(2)}_\mu \{x_n\}. \quad (6)$$

The functions on the right-hand side of (1) — (6) denote the real and imaginary parts of the functions taken on the left.

4. Some formulae involving Kelvin's generalised functions

From § 1 (1 — 4) it can be seen that

$$J_{\mu}^{(1)}\{x_n\, e^{\frac{1}{2}n\pi i}\} = e^{\frac{1}{2}\mu\pi i}\, I_{\mu}^{(2)}\{x_n\},\; J_{\mu}^{(2)}\{x_n\, e^{\frac{1}{2}n\pi i}\} = e^{\frac{1}{2}\mu\pi i}\, I_{\mu}^{(1)}\{x_n\}\,; \quad (1)$$

$$I_{\mu}^{(1)}\{x_n\, e^{\frac{1}{2}n\pi i}\} = e^{\frac{1}{2}\mu\pi i}\, J_{\mu}^{(2)}\{x_n\},\; I_{\mu}^{(2)}\{x_n\, e^{\frac{1}{2}n\pi i}\} = e^{\frac{1}{2}\mu\pi i}\, J_{\mu}^{(1)}\{x_n\}\,; \quad (2)$$

and from § 2 (1, 6)

$$H_{\mu}^{(1),(2)}\{x_n\, e^{\frac{1}{2}n\pi i}\} = \frac{J_{-\mu}^{(2)}\{x_n\, e^{\frac{1}{2}n\pi i}\} - e^{-\mu\pi i}\, J_{\mu}^{(2)}\{x_n\, e^{\frac{1}{2}n\pi i}\}}{i \sin \mu\pi}$$

$$= \frac{e^{-\frac{1}{2}\mu\pi i}\, I_{-\mu}^{(1)}\{x_n\} - e^{-\frac{1}{2}\mu\pi i}\, I_{\mu}^{(1)}\{x_n\}}{i \sin \mu\, \pi}$$

$$= \frac{2}{\pi i}\, e^{-\frac{1}{2}\mu\pi i}\, K_{\mu}^{(1)}\{x_n\}.$$

Therefore

$$K_{\mu}^{(1)}\{x_n\} = \frac{\pi}{2}\, i\, e^{\frac{1}{2}\mu\pi i}\, H_{\mu}^{(1),(2)}\{x_n\, e^{\frac{1}{2}n\pi i}\}. \quad (3)$$

In a similar manner § 2 (1,5) gives

$$K_{\mu}^{(2)}\{x_n\} = \frac{\pi}{2}\, i\, e^{\frac{1}{2}\mu\pi i}\, H_{\mu}^{(1),(1)}\{x_n\, e^{\frac{1}{2}n\pi i}\}. \quad (4)$$

From (4) and the first part of § 3 (5) it follows that

$$H_{\mu}^{(1),(1)}\{x_n\, e^{\frac{3}{4}n\pi i}\} = \frac{2}{\pi i}\, e^{-\frac{1}{2}\mu\pi i}\, K_{\mu}^{(2)}\{x_n\, e^{\frac{1}{4}n\pi i}\},$$

so that

$$\frac{\pi}{2}\, i\, (\mathrm{her}_{\mu}^{(1),(1)}\{x_n\} + i\, \mathrm{hei}_{\mu}^{(1),(1)}\{x_n\})\, (\cos \frac{1}{2}\, \mu\pi + i \sin \frac{1}{2}\, \mu\, \pi) =$$

$$\mathrm{ker}_{\mu}^{(2)}\{x_n\} + i\, \mathrm{kei}_{\mu}^{(2)}\{x_n\}.$$

Therefore

$$\mathrm{ker}_{\mu}^{(2)}\{x_n\} = -\frac{\pi}{2}\, (\sin \frac{1}{2}\, \mu\pi\, \mathrm{her}_{\mu}^{(1),(1)}\{x_n\} + \cos \frac{1}{2}\, \mu\pi\, \mathrm{hei}_{\mu}^{(1),(1)}\{x_n\}), \quad (5)$$

$$\mathrm{kei}_\mu^{(2)}\{x_n\} = \frac{\pi}{2}(\cos\frac{1}{2}\mu\pi\,\mathrm{her}_\mu^{(1),(1)}\{x\} - \sin\frac{1}{2}\mu\pi\,\mathrm{hei}_\mu^{(1),(1)}\{x_n\}). \quad (6)$$

Similarly by considering $H_\mu^{(1),(2)}\ \{x_n\ e^{\frac{3}{4}n\pi i}\}$ and using (3), it is obtained that

$$\mathrm{ker}_\mu^{(1)}\{x\} = -\frac{\pi}{2}(\sin\frac{1}{2}\mu\pi\,\mathrm{her}_\mu^{(1),(2)}\{x_n\} + \cos\frac{1}{2}\mu\pi\,\mathrm{hei}_\mu^{(1),(2)}\{x_n\}), \quad (7)$$

$$\mathrm{kei}_\mu^{(1)}\{x_n\} = \frac{\pi}{2}(\cos\frac{1}{2}\mu\pi\,\mathrm{her}_\mu^{(1),(2)}\{x\} - \sin\frac{1}{2}\mu\pi\,\mathrm{hei}_\mu^{(1),(2)}\{x_n\}). \quad (8)$$

By § 2 (1,5)

$$J_{-\mu}^{(1)}\{(-1)^{n+1}x_n\,e^{\frac{3}{4}n\pi i}\} = e^{-\mu\pi i}\,J_\mu^{(1)}\{x_n\,e^{\frac{3}{4}n\pi i}\} + i\sin\mu\pi\,H_\mu^{(1),(1)}\{x_n\,e^{\frac{3}{4}n\pi i}\},$$

and, therefore

$$\mathrm{ber}_{-\mu}^{(1)}\{(-1)^{n+1}x_n\} + i\,\mathrm{bei}_{-\mu}^{(1)}\{(-1)^{n+1}x_n\} = (\cos\mu\pi - i\sin\mu\pi)(\mathrm{ber}_\mu^{(1)}\{x_n\} +$$

$$+ i\,\mathrm{bei}_\mu^{(1)}(\{x_n\}) + i\sin\mu\pi\,(\mathrm{her}_\mu^{(1),(1)}\{x_n\} + i\,\mathrm{hei}_\mu^{(1),(1)}\{x_n\}).$$

On equating the real and imaginary parts

$$\mathrm{ber}_{-\mu}^{(1)}\{(-1)^{n+1}x_n\} = \cos\mu\pi\,\mathrm{ber}_\mu^{(1)}\{x_n\} - \sin\mu\pi\,(\mathrm{hei}_\mu^{(1),(1)}\{x_n\} -$$

$$- \mathrm{bei}_\mu^{(1)}\{x_n\}), \quad (9)$$

$$\mathrm{bei}_{-\mu}^{(1)}\{(-1)^{n+1}x_n\} = \cos\mu\pi\,\mathrm{bei}_\mu^{(1)}\{x_n\} + \sin\mu\pi\,(\mathrm{her}_\mu^{(1)}\{x_n\} -$$

$$- \mathrm{ber}_\mu^{(1)}\{x_n\}). \quad (10)$$

Again by § 2 (1 ; 5,6)

$$H_{-\mu}^{(1),(1)}\{x_n\} = e^{\mu\pi i}\,H_\mu^{(1),(1)}\{(-1)^{n+1}x_n\}, \quad (11)$$

$$H_{-\mu}^{(1),(2)}\{x_n\} = e^{\mu\pi i}\,H_\mu^{(1),(2)}\{x_n\}\ ; \quad (12)$$

in (11) on replacing x_n by $(-1)^{n+1}x_n\,e^{\frac{3}{4}n\pi i}$ and using first part of § 3 (5) it is observed that

$$\mathrm{her}_{-\mu}^{(1),(1)}\{(-1)^{n+1}x_n\} + i\,\mathrm{hei}_{-\mu}^{(1),(1)}\{(-1)^{n+1}x_n\} = (\cos\mu\pi + i\sin\mu\pi)$$

$$(\mathrm{her}_\mu^{(1),(1)}\{x_n\} + i\,\mathrm{hei}_\mu^{(1),(1)}\{x_n\}).$$

Therefore, on equating the real and the imaginary parts it is obtained that

$$\mathrm{her}_{-\mu}^{(1),(1)}\ \{(-1)^{n+1}\,x_n\} = \cos\mu\pi\,\mathrm{her}_{\mu}^{(1),(1)}\ \{x_n\} - \sin\mu\pi\ \mathrm{hei}_{\mu}^{(1),(1)}\ \{x_n\}, \quad (13$$

$$\mathrm{hei}_{-\mu}^{(1),(1)}\ \{(-1)^{n+1}\,x_n\} = \sin\mu\pi\,\mathrm{her}_{\mu}^{(1),(1)}\ \{x_n\} + \cos\mu\pi\ \mathrm{hei}_{\mu}^{(1),(1)}\ \{x_n\}. \quad (14)$$

For the second type of functions it is deduced that

$$\mathrm{ber}_{-\mu}^{(2)}\,\{x_n\} = \cos\mu\pi\,\mathrm{ber}_{\mu}^{(2)}\,\{x_n\} - \sin\mu\pi\,(\mathrm{hei}_{\mu}^{(1),(2)}\,\{x_n\} - \mathrm{bei}_{\mu}^{(2)}\,\{x_n\}), \quad (15)$$

$$\mathrm{bei}_{-\mu}^{(2)}\,\{x_n\} = \cos\mu\pi\,\mathrm{bei}_{\mu}^{(2)}\,\{x_n\} + \sin\mu\pi\,(\mathrm{her}_{\mu}^{(1),(2)}\,\{x_n\} - \mathrm{ber}_{\mu}^{(2)}\,\{x_n\}), \quad (16)$$

$$\mathrm{her}_{-\mu}^{(1),(2)}\ \{x_n\} = \cos\ \mu\pi\ \mathrm{her}_{\mu}^{(1),(2)}\ \{x_n\} - \sin\ \mu\pi\ \mathrm{hei}_{\mu}^{(1),(2)}\ \{x_n\}, \quad (17)$$

$$\mathrm{hei}_{-\mu}^{(1),(2)}\ \{x_n\} = \cos\ \mu\pi\ \mathrm{hei}_{\mu}^{(1),(2)}\ \{x_n\} + \sin\ \mu\pi\ \mathrm{her}_{\mu}^{(1),(2)}\ \{x_n\}. \quad (18)$$

Similarly, various formulae for different signs of the arguments $x_1, \ldots, x_n$ and the order μ can be derived.

The recurrence formulae for the above defined functions can easily be constructed by means of the recurrence formulae satisfied by their parent functions.

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REFERENCES

1. Watson, G. N. — *Theory of Bessel Functions*, 2nd ed. (1944), p. 81.

2. Whitehead, C. S. — Quarterly Journal, XLII (1911), pp. 316-342.

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