# OPERATORS OF FRACTIONAL INTEGRATION AND A GENERALISED HANKEL TRANSFORM

by

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1. A generalised HANKEL transform is defined by the relation [3]

$$(1.1)\qquad F(x) = \int_0^\infty G^{21}_{24}\left(xy \,\middle|\, \begin{matrix} k-m-\frac{1}{2}-\frac{\nu}{2},\ -k+m+\frac{1}{2}+\frac{\nu}{2} \\ \frac{\nu}{2},\ \frac{\nu}{2}+2m,\ -\frac{\nu}{2},\ -\frac{\nu}{2}-2m \end{matrix}\right) f(y)\,dy$$

With $k+m=\frac{1}{2}$ this reduces to HANKEL transform in Tricomi's form.

As this in similar to the $X_{\nu,k,m}$ — transform of ROOP NARAIN [8, p. 270] we call $F(x)$ as $X_{\nu,k,m}$ — transform of $f(x)$. When $f(x)=F(x)$ then $f(x)$ shall be called self-reciprocal in $X_{\nu,k,m}$ — transform and we shall say that $f(x)$ is $R_\nu(k, m)$.

In this paper we have applied the ERDELYI-KOBER operators [4, p. 293], [6, p. 193] of fractional integration to develope the theory of the generalised HANKEL transform (1.1).

2. We first determine the conditions under which a function $f(x)$ is self-reciprocal in the transform (1.1). Let $M(s)$ be the Mellin transform of $f(x)$. From (1.1), taking the MELLIN transform and changing the order of integration we get (¹).

$$M(s) = \frac{\Gamma_*(\frac{\nu}{2}+s+m\pm m)\ \Gamma(\frac{3}{2}-k+m+\frac{\nu}{2}-s)}{\Gamma_*(\frac{\nu}{2}-s+1+m\pm m)\ \Gamma(\frac{1}{2}-k+m+\frac{\nu}{2}+s)}\, M(1-s)$$

it being assumed that (1.1) is absolutely convergent and Mellin-transform of $|f(x)|$ exists.

So,

$$M(s) = \frac{\Gamma_*(\frac{\nu}{2}+s+m\pm m)}{\Gamma(\frac{1}{2}-k+m+\frac{\nu}{2}+s)}\,\psi(s) \text{ where } \psi(s)=\psi(1-s)$$

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(¹) For the sake of brevity we denote $\Gamma(a+b).\ \Gamma(a-b)$ by $\Gamma_*(a\pm b)$.

By Mellin Inversion Formula [10, p. 7].

$$(2.1)\qquad f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma_*(\frac{\nu}{2}+s+m\pm m)}{\Gamma(\frac{1}{2}-k+m+\frac{\nu}{2}+s)}\,\psi(s)\,x^{-s}\,ds$$

where $\psi(s)=\psi(1-s)$ and $s\equiv\sigma+it$.
We shall saynthat $f(x)$ belongs to $A(\omega,\alpha)$ where $0<\omega\leq\pi$. $\alpha<\frac{1}{2}$ if $(i)$ it is analytic function of $x=\gamma e^{i\theta}$, regular in the angle defined, by $\gamma>0$, $|\theta|<\omega$ (ii) it is $0(|x|^{-\alpha-\delta})$ for small $x$ and $0\,(|x|^{\alpha-1+\delta})$ for large $x$ for every positive $\delta$ and uniformly in any angle $|\theta|\leq\omega-\epsilon<\omega$. A necessary and sufficient condition that a function $f(x)$ of $A(\omega,\alpha)$ be $R_\nu(k,m)$ is that it should be of the form of (2.1) where $\psi(s)$ is regular, satisfies relation $\psi(s)=\psi(1-s)$ in the strip $\alpha<\sigma<1-\alpha$ and is $0\,[e^{(\frac{\pi}{2}-\omega+\varepsilon)|\delta|}]$ for every positive $\varepsilon$ and uniformly in any strip interior to $\alpha<\sigma<1-\alpha$ and $c$ is any value of $\sigma$ in $\alpha<\sigma<1-\alpha$.

We now give a theorem of generalised Hankel transform which transforms one $R_\nu(k,m)$ function into another $R_\nu(k,m)$. Srivastava has given such theorem for $\omega_{\mu,\nu}$ -transform [9, p. 58], [7, p. 826].

Theorem 1 : If $I^+_{\eta,\alpha}$ and $K^-_{\eta,\alpha}$ belong to $L_2$ and $(I^+_{\eta,\alpha})^m(K^-_{\eta,\alpha})^n$ stands for the operators performed $m$ and $n$ times, in any order, then the operator $(I^+_{\eta\alpha,})^m(K^-_{\eta\alpha,})^n+(I^+_{\eta,\alpha})^n(K^-_{\eta,\alpha})^m$ transforms an $R_\nu(k,m)$ into another $R_\gamma(k,m)$ function. Similar property is obeyed by the operator $(I^-_{\eta,\alpha})^m(K^+_{\eta,\alpha})^n+(I^-_{\eta,\alpha})^n(K^+_{\eta,\alpha})^m$.

*Proof :* Let $f(x)$ belonging to $L_2$ be an $R_\nu(k,m)$. Then,

$$(2.2)\qquad f(x)=\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}P(\tfrac{1}{2}+it)\,x^{-\frac{1}{2}-it}\,dt$$

where

$$P(\tfrac{1}{2}+it)=\frac{1}{2\pi}\frac{\Gamma_*(\frac{\nu}{2}+m\pm m+\frac{1}{2}+it)}{\Gamma(1-k+m+\frac{\nu}{2}+it)}\,\psi\,(\tfrac{1}{2}+it)$$

with $\psi\,(\frac{1}{2}+it)=\psi(\frac{1}{2}-it)$ and $\psi\,(\frac{1}{2}-it)=0\,[e^{(\frac{\pi}{2}-\omega+\varepsilon)|t|}]$. Hence,

$$I^+_{\eta,\alpha}\,f(x)=\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}P(\tfrac{1}{2}+it)\,x^{-\frac{1}{2}-it}\,\frac{\Gamma(\eta+\frac{1}{2}-it)}{\Gamma(\eta+\alpha+\frac{1}{2}-it)}\,dt;\ R(\eta)>-\tfrac{1}{2},\ (\alpha)\ R>0$$

This is obtained by substituting for $f(u)$ from (2.2) and changing the order of integrations which is valid dne to the absolute convergence of both $u$ and $t$ integrals. Proceding on these lines we get,

$$\{I^{+}_{\eta,\alpha}\}^m\}K^{-}_{\eta,\alpha}\}^n f(x) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} P\left(\tfrac{1}{2}+it\right) x^{-\frac{1}{2}-it} \frac{\{\Gamma(\eta+\frac{1}{2}-it)\}^m \{\Gamma(\eta+\frac{1}{2}+it)\}^n}{\{\Gamma(\eta+\alpha+\frac{1}{2}-it)\}^m \{\Gamma(\eta+\alpha+\frac{1}{2}+it)\}^n} dt$$

Hence,

$$[(I_{\eta,\alpha})^m (K^{-}_{\eta,\alpha})^n + (I^{+}_{\eta,\alpha})^n (K^{-}_{\eta,\alpha})^m f(x) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} P\left(\tfrac{1}{2}+it\right). x^{-\frac{1}{2}-it} \psi_1\left(\tfrac{1}{2}+it\right) dt$$

where $\psi_1(\frac{1}{2}+it)$ is a function satisfying the relation $\psi_1(\frac{1}{2}+it) = \psi_1(\frac{1}{2}-it)$.

Example : We illustrate the use of the theorem, with $m = n = 1$, in using one $R_\nu(k, m)$ function to get another. Consider the function

$$G^{l,\,n}_{p,\,q}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

Taking its Mellin transform [1, p. 337] and then applying the Mellin inversion formula we obtain,

$$G^{l+2,\,n}_{p+1,\,q+2}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p, \frac{1}{2}-k+m+\frac{\nu}{2} \\ \frac{\nu}{2}, \frac{\nu}{2}+2m, b_1, \ldots, b_q \end{matrix}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma_*(\frac{\nu}{2}+s+m\pm m)}{\Gamma(\frac{1}{2}-k+m+\frac{\nu}{2}+s)} \psi(s) x^{-s} ds$$

where $\psi(s) = \dfrac{\prod_{j=1}^{l} \Gamma(b_j+s) \prod_{j=1}^{n} \Gamma(1-a_j-s)}{\prod_{j=l+1}^{q} \Gamma(1-b_j-s) \prod_{j=n+1}^{p} \Gamma(a_j+s)}$. The $G$-function is self-reciprocal if $\psi(s) = \psi(1-s)$, for which $l = n$, $p = q$ and $a_j + b_j = 0$, $j = 1, 2, \ldots, p$.

Hence

$$f(x) = G^{n+2,\,n}_{p+1,\,p+2}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p, \frac{1}{2}-k+m+\frac{\nu}{2} \\ \frac{\nu}{2}, \frac{\nu}{2}+2m, -a_1, \ldots, -a_p \end{matrix}\right)$$

is $R_1(k, m)$. Using the known integrals [2, p. 200, 212] we have

$$K^{-}_{\eta,\alpha}[I^{+}_{\eta,\alpha} f(x)] = G^{n+3,\,n+1}_{p+3,\,p+4}\left(x \,\middle|\, \begin{matrix} -\eta, a_1, \ldots, a_p, \frac{1}{2}-k+m+\frac{\nu}{2}, \eta+\alpha \\ \eta, \frac{\nu}{2}, \frac{\nu}{2}+2m, -a_1, \ldots, -a_p, -\eta-\alpha \end{matrix}\right)$$

$2p < 4n+1$, $R(\eta) > -\frac{1}{2}$, $R(\frac{\nu}{\nu} + m \pm m) > -1$, $R(a_j) < 1$, $j = 1, 2, \ldots, n$;

$0 < R(\alpha) < 1 + R(\eta)$; $R(\alpha) < 1 - R(a_j)$, $< 1$, $j = 1, 2, \ldots, p$.

which by the theorem is another $R_\nu(k, m)$ function.
Putting $n = p = 0$ and using [2, p. 439], we obtain,

$$x^{-\eta-1} E \begin{pmatrix} 2\eta + 1, \eta + 1 + \frac{\nu}{2}, \eta + 2m + 1 + \frac{\nu}{2}, 1 - \alpha ; x \\ \frac{3}{2} - k + m + \frac{\nu}{2} + \eta, 2\eta + \alpha + 1 \end{pmatrix}$$

is $R_\nu(k, m)$. Now with $k + m = \frac{1}{2}$ we have

$$x^{-\eta-1} E \begin{pmatrix} 2\eta + 1, \eta + \frac{\nu}{2} + 1, 1 - \alpha ; x \\ 2\eta + \alpha + 1 \end{pmatrix}$$

as a $R_\nu$ function.

3. In this section we consider the operation changing the pair of functions $f$ and its $x_{\nu,k,m}$ -transform to the second pair of functions, one being the $x_{\nu^1,k^1,m^1}$ -transform of the other. We have the following theorems.

THEOREM 2: If,

$$F = X_{\nu,k,m} f, \tag{3.1}$$

then,

$$\int_0^\infty K(xy)\; F(y)\; dy = X_{\nu^1,k^1,m^1} \left\{ \int_0^\infty k(xy)\; f(y)\; dy \right\} \tag{3.2}$$

provided

$$\begin{aligned} &\Gamma_*(\tfrac{\nu}{2} - it + m \pm m \pm \tfrac{1}{2})\; \Gamma_*(\tfrac{\nu^1}{2} - it + m^1 \pm m^1 + \tfrac{1}{2}) \\ &\times \Gamma(\tfrac{\nu}{2} + 1 - k + m + it)\; \Gamma(\tfrac{\nu^1}{2} + 1 - k^1 + m^1 + it)\; Mt\;(k) \end{aligned} \tag{3.3}$$

$$\begin{aligned} &= \Gamma_*(\tfrac{\nu}{2} + it + m \pm m + \tfrac{1}{2})\; \Gamma_*(\tfrac{\nu^1}{2} + it + m^1 \pm m^1 + \tfrac{1}{2}) \\ &\times \Gamma(\tfrac{\nu}{2} + 1 - k + m - it)\; \Gamma(\tfrac{\nu^1}{2} + 1 - k' + m' - it)\; M_{-t}\,(k) \end{aligned}$$

the integrals on both sides of (3.2) are absolutely convergent, MELLIN transforms [4, p. 298] of |L.H.S.|, |R.H.S.| and $|x_{\nu^1,k^1,m^1}$ -transform of $k_{(xy)}|$ exist, $x_{\nu^1,k^1,m^1}$ -transform of $k(xy)$ exists, $R(\nu^1 + 2m^1 \pm 2m^1 + 1) > 0$, $0 < R(k^1 - m^1 - \frac{\nu'}{2}) < 1$ and $2m^1$ is not an integer or zero.

*Proof:* Taking MELLIN transform of [4, p. 298] both sides of (3.2) we have,

$$(3.4) \qquad \int_0^\infty x^{-\frac{1}{2}+it}\, dx \int_0^\infty K(xy)\; F(y)\; dy$$

$$= \int_0^\infty x^{-\frac{1}{2}+it}\, dx \int_0^\infty G^{21}_{24}\left(xy \,\middle|\, \begin{matrix} k^1-m^1-\frac{1}{2}-\frac{\nu^1}{2},\ -k^1+m^1+\frac{1}{2}+\frac{\nu^1}{2} \\ \frac{\nu^1}{2}, \frac{\nu^1}{2}+2m^1,\ -\frac{\nu^1}{2},\ -\frac{\nu^1}{2}-2m^1 \end{matrix}\right) dy \int_0^\infty k(zy)\; f(z)\; dz$$

On the L.H.S. we replace $xy$ by $u$ after changing the order of integration which is valid provided the integral on the L.H.S. of (3.2) is absolutely convergent and MELLIN transform of $|\int_0^\infty k(xy)\; F(y)\; dy|$ exists. So L.H.S.

$$= \int_0^\infty F(y)\; dy\; y^{-\frac{1}{2}-it} \int_0^\infty K(u).\; u^{-\frac{1}{2}+it}\, du$$

$$= M_{-t}\,(F)\;\, M_t\,(K)$$

$$= \frac{\Gamma_*\left(\frac{\nu}{2}-it+m\pm m\pm\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}+1-k+m+it\right)}{\Gamma_*\left(\frac{\nu}{2}+it+m\pm m+\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}+1-k+m-it\right)} M_t(f)\, M_t(K) \qquad \text{(A)}$$

on using (3.1).
Also we have from (3.4),

R.H.S.

$$= \int_0^\infty x^{-\frac{1}{2}+it}\, dx \int_0^\infty f(z)\, dz \int_0^\infty G^{21}_{24}\left(xy \,\middle|\, \begin{matrix} k^1-m^1-\frac{1}{2}-\frac{\nu^1}{2},\ -k^1+m^1+\frac{1}{2}+\frac{\nu^1}{2} \\ \frac{\nu^1}{2}, \frac{\nu^1}{2}+2m^1,\ -\frac{\nu^1}{2},\ -\frac{\nu^1}{2}-2m^1 \end{matrix}\right) k(zy)\, dy$$

$$= \int_0^\infty f(z)\, dz \int_0^\infty x^{-\frac{1}{2}+it}\, dx \int_0^\infty G^{21}_{24}\left(xy \,\middle|\, \begin{matrix} k^1-m^1-\frac{1}{2}-\frac{\nu^1}{2},\ -k^1+m^1+\frac{1}{2}+\frac{\nu^1}{2} \\ \frac{\nu^1}{2}, \frac{\nu^1}{2}+2m^1,\ -\frac{\nu^1}{2},\ -\frac{\nu^1}{2}-2m^1 \end{matrix}\right) k(zy)\; dy$$

$$= \int_0^\infty f(z)\, dz \int_0^\infty k(zy)\; dy \int_0^\infty x^{-\frac{1}{2}+it}\, G^{21}_{24}\left(xy \,\middle|\, \begin{matrix} k^1-m^1-\frac{1}{2}-\frac{\nu^1}{2},\ -k^1+m^1+\frac{1}{2}+\frac{\nu^1}{2} \\ \frac{\nu^1}{2}, \frac{\nu^1}{2}+2m,\ -\frac{\nu^1}{2},\ -\frac{\nu^1}{2}-2m^1 \end{matrix}\right) dx.$$

With the help of [2, p. 418] and putting $zy=u$ we obtain, R.H.S.

$$= M_t(f) M_{-t}(k)\, \frac{\Gamma_*\left(\frac{\nu^1}{2}+it+m^1\pm m^1+\frac{1}{2}\right)\Gamma\left(\frac{\nu^1}{2}+1-k^1+m^1-it\right)}{\Gamma_*\left(\frac{\nu^1}{2}-it+m^1\pm m^1+\frac{1}{2}\right)\Gamma\left(\frac{\nu^1}{2}+1-k^1+m^1+it\right)} \qquad \text{(B)}$$

The change in the order of integration is permissible provided $\int_0^\infty k(zy)\, f(z)\, dz$ is absolutely convergent. $X_{\nu^1,k^1,m^1}$ -transform of $k(zy)$ and $|\int_0^\infty k(zy)\, f(z)\, dz|$ exists. MELLIN transform of $|X_{\nu^1,k^1,m^1}$ -transform of $k(zy)|$ exists and $R(\nu^1+2m^1\pm 2m^1+1)>0$, $0<R(k^1-m^1-\frac{\nu^1}{2})<1$. $2m^1$ is not integer or zero. From (A) and (B) we have the relation (3.3) established.

THEOREM 3: If $F = X_{\nu,k,m}\, f$

then,

$$\int_0^\infty \frac{1}{y} K\left(\frac{x}{y}\right) F(y)\, dy = X_{\nu^1,k^1,m^1}\left\{\int_0^\infty \frac{1}{y} k\left(\frac{x}{y}\right) f(y)\, dy\right\} \tag{3.5}$$

provided

$$\begin{aligned}
&\Gamma_*(\tfrac{\nu}{2}+m\pm m+it+\tfrac{1}{2})\ \Gamma_*(\tfrac{\nu^1}{2}+m^1\pm m^1-it+\tfrac{1}{2})\\
&\times\Gamma(\tfrac{\nu}{2}+1-k+m-it)\ \Gamma(\tfrac{\nu^1}{2}+1-k^1+m^1+it)\ M_t(K)\\
&=\Gamma_*(\tfrac{\nu}{2}+m\pm m-it+\tfrac{1}{2})\ \Gamma_*(\tfrac{\nu^1}{2}+m^1\pm m^1+it+\tfrac{1}{2})\\
&\times\Gamma(\tfrac{\nu}{2}+1-k+m+it)\ \Gamma(\tfrac{\nu^1}{2}+1-k^1+m^1-it)\ M_{-1}(k)
\end{aligned} \tag{3.6}$$

the integrals on both sides of (3.5) are absolutely convergent, MELLIN transforms of |L.H.S.), |R.H.S.| and $|X_{\nu^1,k^1,m^1}$ -transform of $k(\frac{x}{y})|$ exist, $X_{\nu^1,k^1,m^1}$ -transform of $k(\frac{x}{y})$ exists, $R(\nu^1+2m^1\pm 2m^1+1)>0$, $0<R(k'-m'-\frac{\nu'}{2})<1$ and $2m'$ is not an integer or zero.
The proof is similar to that of Theorem 2.

4. We now obtain the representations for the kernels used in theorems 2 and 3.
From (3.3),

$$\begin{matrix}K(x)\\ k(x)\end{matrix} = M_x^{-1}\left\{\Gamma_*(\tfrac{\nu^1}{2}+m^1\pm m^1+it+\tfrac{1}{2})\ \Gamma_*(\tfrac{\nu}{2}+m\pm m+it+\tfrac{1}{2})\right. \tag{4.1}$$
$$\left.\Gamma(\tfrac{\nu^1}{2}+1-k^1+m^1-it)\ \Gamma(\tfrac{\nu}{2}+1-k+m-it)\begin{matrix}\Phi(t)\\ \Phi(-t)\end{matrix}\right\}$$

and from (3.6),

$$\begin{matrix}K(x)\\ k(x)\end{matrix} = M_x^{-1}\left\{\Gamma_*(\tfrac{\nu}{2}+m\pm m-it+\tfrac{1}{2})\ \Gamma_*(\tfrac{\nu^1}{2}+m^1\pm m^1+it+\tfrac{1}{2})\right. \tag{4.2}$$
$$\left.\Gamma(\tfrac{\nu}{2}+1-k+m+it)\ \Gamma(\tfrac{\nu^1}{2}+1-k^1+m^1-it)\begin{matrix}\Phi(t)\\ \Phi(-t)\end{matrix}\right\}$$

where $\Phi(t)$ is an arbitrary function such that the R.H.S. of (4.1) and (4.2) exist. Let $\Phi(t) = M_t\, h$ where $h(y)$ is an arbitrary function. Hence, we have from (4.1).

$$\begin{matrix} K(x) \\ k(x) \end{matrix} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma^*\left(\tfrac{\nu^1}{2} + m^1 \pm m^1 + it + \tfrac{1}{2}\right) \Gamma_*\left(\tfrac{\nu}{2} + m \pm m + it + \tfrac{1}{2}\right)$$

$$\Gamma\left(\tfrac{\nu^1}{2} + 1 - k^1 + m^1 - it\right) \Gamma\left(\tfrac{\nu}{2} + 1 - k + m - it\right) x^{-\frac{1}{2}-it} \begin{matrix} M_t h \\ M_{-t} h \end{matrix}\, dt$$

Replacing $y$ by $\frac{1}{y}$ and changing the order of integration, we obtain the representation for Theorem 2,

$$\begin{matrix} K(x) \\ k(x) \end{matrix} = \int_0^{\infty} \frac{1}{y} \begin{matrix} h(\frac{1}{y}) \\ h(y) \end{matrix}\, \lambda_1(xy)\, dy$$

where

$$\lambda_1(x) = \frac{1}{2\pi} \int_0^{\infty} \Gamma_*\left(\tfrac{\nu^1}{2} + m^1 \pm m^1 + it + \tfrac{1}{2}\right) \Gamma_*\left(\tfrac{\nu}{2} + m \pm m + it + \tfrac{1}{2}\right)$$

$$\Gamma\left(\tfrac{\nu^1}{2} + 1 - k^1 + m^1 - it\right) \Gamma\left(\tfrac{\nu}{2} + 1 - k + m - it\right) x^{-\frac{1}{2}-it}\, dt$$

and $R(\frac{\nu}{2}) - \frac{1}{2}$, $R(\frac{\nu^1}{2}) - \frac{1}{2}$, $R(\frac{\nu^1}{2} - k^1 + m^1)$, $R(\frac{\nu}{2} - k + m)$ are not negative integers, $2m$, $2m^1$ are not integral or zero.
Similarly for Theorem 3 we get from (4.2)

$$\begin{matrix} K(x) \\ k(x) \end{matrix} = \int_0^{\infty} \frac{1}{y} \begin{matrix} h(\frac{1}{y}) \\ h(y) \end{matrix}\, \lambda_2(xy)\, dy$$

where

$$\lambda_2(x) = \frac{1}{2\pi} \int_0^{\infty} \Gamma_*\left(\tfrac{\nu}{2} + m \pm m - it + \tfrac{1}{2}\right) \Gamma_*\left(\tfrac{\nu^1}{2} + m^1 \pm m^1 + it + \tfrac{1}{2}\right)$$

$$\Gamma\left(\tfrac{\nu}{2} + 1 - k + m + it\right) \Gamma\left(\tfrac{\nu^1}{2} + 1 - k' + m' - it\right) x^{-\frac{1}{2}-it}\, dt$$

and $R(\frac{\nu}{2}) - \frac{1}{2}$, $R(\frac{\nu^1}{2}) - \frac{1}{2}$, $R(\frac{\nu^1}{2} - k + m)$, $R(\frac{\nu}{2} - k + m)$ are not negative integers ; $2m$, $2m^1$ are not integral or zero
We obtain other representations of the kernels of Theorems 2 and 3 using an operator $T_{\eta,\alpha}$ defined by ERDELYI [5, p. 113] as

$$T_{\eta,\alpha}\, f(x) = l.\,i.\,m.\; x^{-\frac{\alpha}{2}} \int_0^{\infty} y^{-\frac{\alpha}{2}} J_{2\eta+\alpha}\left(2\sqrt{\tfrac{x}{y}}\right) f(y)\, dy$$

with $R(\eta) > -\frac{1}{2}$, $R(\alpha) > \frac{1}{2}$.

So,

$$M_t\,(T_{\eta,\alpha}\, f) = \frac{\Gamma(\eta + \frac{1}{2} + it)}{\Gamma(\eta + \alpha + \frac{1}{2} - it)}\, M_t\, f$$

Hence we have from (3.3) and (3.6),

$$\begin{matrix} K(x) \\ k(x) \end{matrix} = T_{\frac{\nu}{2},\frac{\mu}{2}} \left[ K_{\frac{\mu}{2},\frac{\nu}{2}} \left\{ I_{\frac{\nu}{2}+\frac{1}{2}-k+m,k+m-\frac{1}{2}} \left( I_{\frac{\mu}{2}+\frac{1}{2}-k+m,k+m-\frac{1}{2}} \begin{matrix} h(x) \\ \frac{1}{x}\, h(\frac{1}{x}) \end{matrix} \right) \right\} \right]$$

and

$$\begin{matrix} K(x) \\ k(x) \end{matrix} = I_{\frac{\mu}{2}+2m,\frac{\nu}{2}-\frac{\mu}{2}-2m} \left[ K_{\frac{\nu}{2}+2m,\frac{\mu}{2}-\frac{\nu}{2}-2m} \left\{ K_{\frac{\mu}{2}+\frac{1}{2}-k+m,\frac{\nu}{2}} \left( I_{\frac{\nu}{2}+\frac{1}{2}-k+m,\frac{\mu}{2}} \begin{matrix} h(x) \\ \frac{1}{x}\, h(\frac{1}{x}) \end{matrix} \right) \right\} \right]$$

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