CERTAIN PROPERTIES OF VARMA TRANSFORM INVOLVING WHITTAKER FUNCTIONS

by

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1. INTRODUCTION. VARMA [6, p. 209] has given a generalization of the classical LAPLACE's integral

$$\Phi(p) = p \int_0^\infty e^{-pt}\, h(t)\, dt, \tag{1.1}$$

in the form

$$\Phi(p) = p \int_0^\infty e^{-\frac{1}{2}pt} (pt)^{m-1/2}\, W_{k,m}(pt)\, h(t)\, dt, \tag{1.2}$$

which reduces to (1.1) when $k = -m + {}^1/_2$.

In this paper we have obtained certain new properties of VARMA transform which enable us to evaluate certain infinite integrals involving special functions occuring in Applied Mathematics. These properties have been given in the form of three theorems. The results obtained are very interesting and are believed to be new.

In what follows the notations

$$\Phi(p) \doteqdot h(t) \quad \text{and} \quad \Phi(p) \underset{k,\,m}{\overset{V}{=\!=}} h(t)$$

will be used to denote (1.1) and (1.2) respectively.

2. THEOREM I. If

$$\Phi(p) \doteqdot h(t), \tag{2.1}$$

and

$$\Psi(p, k, m, a) \underset{k,\,m}{\overset{V}{=\!=}} K_{2m}(at^{1/2})\, t^{-m}\, h(t),$$

then

$$\Psi(p, k, m, a) = \frac{p^{m+1/2}}{a} \int_0^\infty t^{-k} (t+p)^{k-1}$$

$$\times \exp\left\{ -\frac{2t+p}{8t(t+p)} a^2 \right\} W_{k,m} \left[\frac{a^2 p}{4t(t+p)} \right] \Phi(t+p)\, dt, \quad (2.2)$$

provided that $h(t)$ is independent of a, the integral in (2.2) is convergent, $R(p) > 0$, $R(a^2) > 0$, the VARMA transforms of $|h(t)|$ and $|t^{-m} K_{2m}(at^{1/2})\, h(t)|$ exist.

Proof. Applying the PARSEVAL-GOLDSTEIN theorem of the operational calculus to (2.1) and [3]

$$p^{\frac{1}{2}} K_{2m}(ap^{1/2})\, e^{\frac{1}{2}pb}\, W_{k,m}(pb)$$

$$\doteqdot \frac{1}{a} t^{-k} (t+b)^k \exp\left\{ -\frac{(2t+b)}{8t(t+b)} a^2 \right\}$$

$$\times W_{k,m}\left[\frac{a^2 b}{4t(t+b)} \right], \quad (2.3)$$

where $R(a^2) > 0$, $R(p) > 0$, we easily obtain

$$\int_0^\infty t^{-\frac{1}{2}} K_{2m}(at^{\frac{1}{2}})\, e^{-\frac{1}{2}bt}\, W_{k,m}(bt)\, h(t)\, dt$$

$$= \frac{1}{a} \int_0^\infty t^{-k} (t+b)^{k-1} \exp\left\{ -\frac{(2t+b)\, a^2}{8t(t+b)} \right\} W_{k,m}\left[\frac{a^2 b}{4t(t+b)} \right]$$

$$\times \Phi(t+b)\, dt, \quad (2.4)$$

On interpreting the l.h.s of (2.4) with the help of (1.2) we arrive at the result.

EXAMPLE 1. If we take

$$\left. \begin{aligned} h(t) &= t^{\varrho} \\ &\doteqdot \Gamma(\varrho+1)\, p^{-\varrho} = \Phi(p) \end{aligned} \right\} \quad (2.5)$$

where $R(\varrho + 1) > 0$, $R(p) > 0$ we then have [5, p. 110]

$$t^{-m} K_{2m}(at^{1/2})\, h(t) = t^{\varrho-m} K_{2m}(at^{1/2})$$

$$\underset{k,\,m}{\overset{V}{=\!=}} \tfrac{1}{2} p^{m}\, G^{2,2}_{2,3}\left(\frac{a^2}{4p}\middle| \begin{matrix} -\varrho - m,\; m - \varrho \\ m,\; -m,\; k - \varrho - {}^1/_2 \end{matrix}\right)$$

$$= \Psi(p, k, m, a),$$

where $R(\varrho \pm 2m + 1) > 0$, $R(p) > 0$ and G-denotes the MEIJER'S G-function.

Applying (2.2) it is found that

$$\int_0^{\infty} t^{-k}(t+p)^{k-\varrho-1} \exp\left\{-\frac{(2t+p)\,a^2}{8t\,(t+p)}\right\}$$

$$\times\, W_{k,m}\left[\frac{a^2\, p}{4t\,(t+p)}\right] dt$$

$$= \frac{a}{2p^{\varrho+1/2}\,\Gamma(\varrho+1)}\, G^{2,2}_{2,3}\left(\frac{a^2}{4p}\middle| \begin{matrix} -\varrho - m,\; m - \varrho) \\ m,\; -m,\; k - \varrho - {}^1/_2 \end{matrix}\right), \tag{2.6}$$

where $R(\varrho \pm 2m + 1) > 0$, $|\arg p| < \pi$, $R(a^2) > 0$.

3. With the help of the operational relation [3]

$$p^{\frac{1}{2}}\, I_{2m}(ap^{\frac{1}{2}})\, e^{\frac{1}{2}pb}\, W_{k,m}(pb)$$

$$\doteqdot \frac{2t^{-k}\,(t+b)^k}{a\,\Gamma({}^1/_2 - k - m)\,\Gamma(2m+1)} \exp\left\{-\frac{(2t+b)\,a^2}{8t\,(t+b)}\right\}$$

$$\times\, M_{k,m}\left[\frac{a^2\, b}{4t\,(t+b)}\right], \tag{3.1}$$

where $R(a^2) > 0$, $R(p) > 0$ and following a similar procedure, we obtain

THEOREM 2. If

$$\Phi(p) \doteqdot h(t)$$

and

$$\Psi(p, k, m, a) \underset{k,\,m}{\overset{V}{=\!=}} t^{-m} I_{2m}(at^{1/2})\; h(t),$$

then

$$\Psi(p, k, m, a) = \frac{2p^{m+1/2}}{a\,\Gamma(\tfrac{1}{2} - k - m)\,\Gamma(2m+1)}$$

$$\times \int_0^\infty t^{-k}(t+p)^{k-1} \exp\left\{-\frac{(2t+p)\,a^2}{8t\,(t+p)}\right\} M_{k,m}\left[\frac{a^2 p}{4t\,(t+p)}\right]$$

$$\times\ \Phi(p+t)\,dt, \tag{3.2}$$

provided that the integral in (3.2) is convergent, $h(t)$ is independent of a, $R(a^2) > 0$, $R(p) > 0$ and the VARMA transforms of $|h(t)|$ and $|t^{-m} I_{2m}(at^{\frac{1}{2}})\,h(t)|$ exist.

EXAMPLE 2. Taking (2.5), we have [**1**, p. 85, eq. 16]

$$t^{-m} I_{2m}(at^{1/2})\,h(t) = t^{\varrho-m} I_{2m}(at^{1/2})$$

$$\underset{k,\,m}{\overset{V}{=}} \frac{a^{2m}\,\Gamma(\varrho+2m+1)\,\Gamma(\varrho+1)}{2^{2m}\,p^{\varrho}\,\Gamma(1+2m)\,\Gamma(\tfrac{1}{2}-k+m+\varrho)}$$

$$\times\ {}_2F_2\left(\varrho+2m+1,\ \varrho+1;\ 2m+1.\ \varrho+m-k+{}^3/_2;\ \frac{a^2}{4p}\right)$$

$$= \Psi(p, k, m, a)$$

where $R(\varrho + m \pm m + 1) > 0$, $R(p) > 0$.

From (3.2) it is easily seen that

$$\int_0^\infty t^{-k}(t+p)^{k-\varrho-1} \exp\left\{-\frac{(2t+p)\,a^2}{8t\,(t+p)}\right\}$$

$$\times\ M_{k,m}\left[\frac{a^2\,p}{4t\,(t+p)}\right] dt$$

$$= \frac{a^{2m+1}}{2^{2m+1}} \frac{\Gamma(\varrho+2m+1)\ \Gamma(\tfrac{1}{2}-k+m)}{\Gamma({}^3/_2-k+m+\varrho)}$$

$$\times\ {}_2F_2\left(\varrho+2m+1,\ \varrho+1;\ 2m+1,\ \varrho+m-k+{}^3/_2;\ {a^2}\Big/{4p}\right), \tag{3.3}$$

where $R(\varrho + 2m + 1) > 0$, $|\arg p| < \pi$, $R(a^2) > 0$, $\left|\frac{a^2}{4p}\right| < 1$.

4. THEOREM 3. If

$$\Phi(p) \underset{k,\,m}{\overset{V}{=\!=}} h(t)$$

and

$$\Psi(p) \frac{V}{k - {}^1\!/_4,\; m + {}^1\!/_4} t^{-\frac{1}{2}} h(t),$$

then

$$\int_0^\infty x^{1/2} (ax^2 + bx + c)^{-1} \, \Phi\left(\frac{ax^2 + bx + c}{x}\right) dx$$

$$= \frac{\pi^{\frac{1}{2}}}{a^{1/2}\,(b + 2\sqrt{(ac)})} \Psi(b + 2\sqrt{(ca)}), \tag{4.1}$$

where $R(a) > 0$, $R(c) > 0$ and

$$\int_0^\infty \cosh \tfrac{1}{2}\theta \; (\alpha + \beta \cosh \theta)^{-1} \;\; \phi(\alpha + \beta \cosh \theta)\; d\theta$$

$$= \frac{\pi^{\frac{1}{2}}\,\Psi(\alpha + \beta)}{\sqrt{(2\beta)}\;(\alpha + \beta)}, \tag{4.2}$$

where $R(\beta) > 0$, provided that the integrals involved are absolutel y convergent.

Proof. From (1.2) it follows that

$$\int_0^\infty x^{1/2} (ax^2 + bx + c)^{-1} \, \Phi\left(\frac{ax^2 + bx + c}{x}\right) dx$$

$$= \int_0^\infty x^{-1/2} \left[\int_0^\infty t^{m-1/2} \left(\frac{ax^2 + bx + c}{x}\right)^{m-1/2} \right.$$

$$\left. \times \exp\left\{-\frac{(ax^2 + bx + c)\,t}{2x}\right\} W_{k,m}\left\{\frac{t\,(ax^2 + bx + c)}{x}\right\} g(t)\, dt \right] dx$$

$$= \int_0^\infty t^{m-1/2} g(t) \left[\int_0^\infty x^{-1/2} \left(\frac{ax^2 + bx + c}{x}\right)^{m-1/2} \right.$$

$$\left. \times \exp\left\{-\frac{(ax^2 + bx + c)\,t}{2x}\right\} W_{k,m}\left\{\frac{t\,(ax^2 + bx + c)}{x}\right\} dx \right] dt$$

On evaluating the x-integral by author's formula [**4**, p. 663]

$$\int_0^\infty x^{-\frac{1}{2}} \left(\frac{ax^2+bx+c}{x}\right)^{m-1/2} \exp\left(-\frac{ax^2+bx+c}{2x}\right) W_{k,m}\left(\frac{ax^2+bx+c}{x}\right) dx$$

$$= \left(\frac{\pi}{a}\right)^{\frac{1}{2}} (b+2\sqrt{ca})^{m-1/2} \exp\{-\tfrac{1}{2}(b+2\sqrt{ac})\}\, W_{k-\frac{1}{2},m+\frac{1}{2}}(b+2\sqrt{ac}), \quad (4.3)$$

where $R(a) > 0$, $R(c) > 0$; we obtain (4.1)

If we put $a = c = {}^{\beta}/_{2}$, $b = \alpha$ and use the substitution $x = e^{\theta}$ (4.1) then reduces to (4.2).

The interchange of the order of integration can easily be seen to be permissible under the conditions stated with the theorem.

The theorem is useful in obtaining the integral representations of the VARMA transforms of certain functions. This is shown by the following example.

EXAMPLE 3. If we take

$$g(t) = t^{\gamma} E(\alpha_3, \alpha_4, \ldots, \alpha_r : \beta_2, \ldots, \beta_s : {}^{1}/_{t})$$

then [**2**, p. 133]

$$\Phi(p) = p^{-\gamma} E(\gamma+1, \gamma+2m+1, \alpha_r, \ldots, \alpha_r : \gamma+m-k+{}^{1}/_{2}, \beta_2., \beta_s : p)$$

where $R(\gamma + 2m + 1) > 0$, $R(\gamma + 1) > 0$ and E-denotes MACROBERT'S E-function.

Applying (4.1) and (4.2), replacing $\gamma + 1$ by α_1, $\gamma + 2m + 1$ by α_2 $\gamma + m - k + {}^{1}/_{2}$ by β_1, it is found that

$$\int_0^\infty x^{-1/2} \left(\frac{ax^2+bx+c}{x}\right)^{-\alpha_1} E\left[r;\alpha_i : s;\beta_j : \frac{ax^2+bx+c}{x}\right] dx$$

$$= \sqrt{\frac{\pi}{a}}\, (hb+2\sqrt{ca})^{1/2-\alpha_1} E(\alpha_1 - {}^{1}/_{2}, \alpha_2, \ldots, \alpha_r : \beta_1, \ldots, \beta_s : b+2\sqrt{ca}), \quad (4.4)$$

where $R(\alpha_1) > {}^{1}/_{2}$, $R(a) > 0$, $R(c) > 0$ and

$$\int_0^\infty \cosh \tfrac{1}{2}\theta\, (\alpha + \beta \cosh \theta)^{-\alpha_1} E(r;\alpha_i : s;\beta_j : \alpha + \beta \cosh \theta)\, d\theta$$

$$= \sqrt{\frac{\pi}{2\beta}} (\alpha+\beta)^{\frac{1}{2}-\alpha_1} E(\alpha_1 - {}^{1}/_{2}, \alpha_2, \ldots, \alpha_r : \beta_1, \ldots, \beta_s : \alpha+\beta), \quad (4.5)$$

where $R(\alpha_1) > {}^{1}/_{2}$ and $R(\beta) > 0$.

The following particular cases of (4.4) and (4.5) are worth mentioning.

(*i*) Putting $r = 2,\ s = 0,\ \alpha_1 = \frac{1}{2} - \lambda - \mu,\ \alpha_2 = \frac{1}{2} - \lambda + \mu$ and using the well- known property of the E-function

$$E\left(\tfrac{1}{2}-\lambda-\mu, \tfrac{1}{2}-\lambda+\mu :: x\right) = \Gamma\left(\tfrac{1}{2}-\lambda-\mu\right) \Gamma\left(\tfrac{1}{2}-\lambda+\mu\right) e^{\frac{1}{2}x} W_{,\mu}(x) \quad (4.6)$$

it is seen that

$$\int_0^\infty x^{-\frac{1}{2}} \left(\frac{ax^2+bx+c}{x}\right)^{\mu-1/2} \exp\left(\frac{ax^2+bx+c}{2x}\right) W_{\lambda,\mu}\left(\frac{ax^2+bx+c}{x}\right) dx$$

$$= \frac{\Gamma(-\lambda-\mu)\,\pi^{\frac{1}{2}}}{\Gamma(1/2-\lambda-\mu)\,a^{\frac{1}{2}}} (b+2\sqrt{ca})^{\mu-1/4} \exp\left(\frac{b+2\sqrt{ca}}{2}\right) W_{\lambda+1/4,\mu+1/4}(b+2\sqrt{ca}), \quad (4.7)$$

where $R(\lambda+\mu) < 0,\ R(a) > 0,\ R(c) > 0$ and

$$\int_0^\infty \cosh\tfrac{1}{2}\theta\,(\alpha+\beta\cosh\theta)^{\mu-1/2} \exp\left(\frac{\alpha+\beta\cosh\theta}{2}\right) W_{\lambda,\mu}(\alpha+\beta\cosh\theta)\,d\theta$$

$$= \frac{\Gamma(-\lambda-\mu)\,\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2}-\lambda-\mu)\,2^{\frac{1}{2}}\beta^{\frac{1}{2}}} (\alpha+\beta)^{\mu-1/4} \exp\left(\frac{\alpha+\beta}{2}\right) W_{\lambda+1/4,\mu+1/4}(\alpha+\beta), \quad (4.8)$$

where $R(\lambda+\mu) < 0,\ R(\beta) > 0$.

(ii) On the other hand if we put $r = 4,\ s = 1,\ \alpha_1 = \frac{1}{2} - \lambda + \mu$ $\alpha_2 = \frac{1}{2} - \lambda - \mu,\ \alpha_3 = \frac{1}{2} - \lambda,\ \alpha_4 = 1 - \lambda,\ \beta_1 = 1 - 2\lambda$ then by virtue of the following property of the E-function

$$E\left(\tfrac{1}{2}-\lambda+\mu, \tfrac{1}{2}-\lambda-\mu, \tfrac{1}{2}-\lambda, 1-\lambda : 1-2\lambda : z\right)$$

$$= \Gamma(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}-\lambda+\mu)\,\Gamma(\tfrac{1}{2}-\lambda-\mu) z^{-\lambda} W_{\lambda,\mu}(2iz^{1/2}) W_{\lambda,\mu}(-2iz^{1/2}), \quad (4.9)$$

it is found that

$$\int_0^\infty x^{-\frac{1}{2}} \left(\frac{ax^2+bx+c}{x}\right)^{-\mu-\frac{1}{2}} W_{\lambda,\mu}\left\{2i\sqrt{\left(\frac{ax^2+bx+c}{x}\right)}\right\} W_{\lambda,\mu}\left\{-2i\sqrt{\left(\frac{ax^2+bx+c}{x}\right)}\right\} dx$$

$$= \frac{(b+2\sqrt{ca})^{\lambda-\mu}}{a^{\frac{1}{2}}\,\Gamma(\frac{1}{2}-\lambda+\mu)\,\Gamma(\frac{1}{2}-\lambda-\mu)} E\left(-\lambda+\mu, \tfrac{1}{2}-\lambda-\mu, \tfrac{1}{2}-\lambda, 1-\lambda : 1-2\lambda : b+2\sqrt{ca}\right), \quad (4.10)$$

where $R(\lambda - \mu) > 0$, $R(a) > 0$, $R(c) > 0$ and

$$\int_0^\infty \cosh \tfrac{1}{2}\theta\, (\alpha + \beta \cosh \theta)^{-\mu-\frac{1}{2}}\, W_{\lambda,\mu}(2i\sqrt{\alpha+\beta\cosh\theta})\, W_{\lambda,\mu}(-2i\sqrt{\alpha+\beta\cosh\theta})\, d\theta$$

$$= \frac{(\alpha+\beta)^{\lambda-\mu}}{(2\beta)^{\frac{1}{2}}\, \Gamma(\frac{1}{2}-\lambda+\mu)\, \Gamma(\frac{1}{2}-\lambda-\mu)}$$

$$\times E(\mu-\lambda, \tfrac{1}{2}-\lambda-\mu, \tfrac{1}{2}-\lambda, 1-\lambda; 1-2\lambda : \alpha+\beta), \quad (4.11)$$

where $R(\lambda - \mu) > 0$ and $R(\beta) > 0$.

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