ON THE ORTHOGONALITY OF SOME SYSTEMS OF POLYNOMIALS

by

W. A. AL-SALAM

1. INTRODUCTION. Consider the BERNOULLI and EULER polynomials defined by means of

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \tag{1.1}$$

$$\frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} . \tag{1.2}$$

We have $B_n(o) = B_n$, the BERNOULLI numbers and $2^n E_n(o) = C_n$ the tangent coefficients.

TOUCHARD [6] defined a set of polynomials $\Omega_n(z)$ so that we have the symbolic orthogonality

$$B^m \Omega_n(B) = K_n \delta_{nm} \quad (o \leq m \leq n), \tag{1.3}$$

where

$$K_n = \frac{(-1)^n [n!]^4}{2^n (2n+1)(1, 3, 5 \ldots (2n-1))^2} .$$

It is understood in (1.3) that after expansion B^k is to be replaced by B_k.

TOUCHARD also indicated that (1.3) leads to actual orthogonality. He proved

$$-\frac{\pi i}{2} \int_{c-\infty i}^{c+\infty i} \frac{\Omega_n(z)\ \Omega_m(z)}{\sin^2 \pi z} dz = 2^n K_n \delta_{nm} \tag{1.4}$$

where $-1 < c < 0$.

. . . .

Supported in part by National Science Foundation grant GP-1593.

CARLITZ [2] generalized (1.3) and (1.4) in the following manner. Define

$$\beta_n = \beta_n(\lambda) = \frac{B_{n+1}(\lambda) - B_{n+1}}{(n+1)\lambda}$$

(obviously $\beta_n(o) = B_n$) and consider the polynomials defined by

$$(1.5) \qquad \Omega_n^{(\lambda)}(z) = (-2)^n (\lambda+1)_n \binom{2n}{n}^{-1} F_n^{\lambda}(1-\lambda+2z)$$

where $F_n^{\lambda}(z)$ is the PASTERNACK [5] polynomial. He proved

$$(1.6) \qquad \beta^r \, \Omega_n^{(\lambda)}(\beta) = K_n^{(\lambda)} \, \delta_{rn} \quad (0 \leq r \leq n)$$

where

$$K_n^{(\lambda)} = \frac{(-1)^n (n!)^2 (1+\lambda)_n (1-\lambda)_n}{(2n+1)\, 2^n \{1, 3, 5 \ldots (2n-1)\}^2}.$$

We have further

$$(1.7) \qquad \frac{\sin \pi\lambda}{2i\lambda} \int_{c-\infty i}^{c+\infty i} \frac{\Omega_n^{(\lambda)}(z)\, \Omega_m^{(\lambda)}(z)}{\sin \pi z \sin \pi(z-\lambda)} dz = 2^n K_n^{(\lambda)} \delta_{nm}$$

where $-1 < c < 0$, $0 \leq \lambda < 1$.

CARLITZ also proved

$$(1.8) \qquad (\tfrac{1}{2}C)^m A_n(\tfrac{1}{2}C) = (-2)^{-n} S_{mn} \quad (0 \leq m \leq n)$$

where

$$\frac{(1+t)^z}{(1-t)^{z+1}} = \sum_{n=0}^{\infty} A_n(z)\, t^n.$$

In this note we shall give different proofs of (1.6), (1.7) and (1.8) making use of known facts in the theory of continued fractions. This method has the advantage that given the moments $\beta_n(\lambda)$ the polynomials of (1.5) and the weight function are obtained in a natural way.

2. *Proof of* (1.6). Let $\Psi(z)$ be the logarithmic derivation of the gamma function of z. We have the asymptotic formula [4, 106]

$$\Psi(z+h) \sim \log z - \sum_{n=1}^{\infty} (-1)^n \frac{B_n(h)}{nz^n}. \quad |\arg z| < \pi.$$

From this formula and familiar properties of the BERNOULLI polynomials we can show that

$$\frac{1}{h}\left\{\Psi(z+1)-\Psi(z+1-h)\right\}\sim\sum_{n=0}^{\infty}\frac{\beta_n(h)}{z^{n+1}}. \tag{2.1}$$

This formula can be proved directly from the generating function of $\beta_n(h)$,

$$\sum_{n=0}^{\infty}\beta_n(h)\frac{t^n}{n!}=\frac{1}{h}\frac{1-e^{th}}{1-e^t}.$$

Now consider the continued fraction [7, p. 372]

$$\frac{1}{2}\left\{\Psi(z+k)-\Psi(z+1-k)\right\}=\frac{k-\frac{1}{2}}{z+}\,\frac{a_1}{z+}\,\frac{a_2}{z+}\cdots \tag{2.2}$$

where

$$a_n=\frac{n^2(n+1-2k)(n-1+2k)}{4(2n-1)(2n+1)}.$$

From it follows, after some elementary transformation, that the continued fraction associated with the series in (2.1) is given by

(2.3)

$$\frac{1}{h}\left\{\Psi(z+1)-\Psi(z+1-h)\right\}=\frac{2}{2z+1-h+}\,\frac{c_1}{2z+1-h+}\,\frac{c_2}{2z+1-h+}\cdots$$

where

$$C_n=\frac{n^2(n^2-h^2)}{4n^2-1}. \tag{2.4}$$

Thus the $\beta_n(h)$ are the moments of the J-fraction (2.3) which is uniformly convergent in every closed domain not including $z=0, -1, -2, \ldots$

Consequently we have the symbolic orthogonality

$$\begin{aligned}\beta^r\,\Phi_n(\beta)&=0 \qquad (r=0,1,\ldots,n-1),\\ &\neq 0 \qquad (r=n),\end{aligned} \tag{2.5}$$

where ϕ_n are the denominators of the convergents of the continued fraction (2.3). They satisfy the recurrence relation

$$\phi_{n+1}(z)=(2z+1-h)\,\phi_n(z)+c_n\,\phi_{n-1}(z).$$

If we compare this recurrence relation with that satisfied by the PASTERNACK polynomials

$$(n+1)\,(n+m+1)\,F_{n+1}^{m}(z) = -\,(2n+1)\,z\,F_{n}^{m}(z) + n\,(n-m)\,F_{n-1}^{m}(z)$$

we see that

$$\phi_n(z) = \Omega_n^{(\lambda)}(z).$$

This completes the proof of (1.6).

3. *Proof of* (1.7). To prove (1.7) we begin by changing (2.3) into a positive definite J-fraction. We get

$$(3.1)\quad \frac{i}{2h}\left\{\Psi\left(\tfrac{1}{2}zi+\tfrac{1}{2}+\tfrac{1}{2}h\right)-\Psi\left(\tfrac{1}{2}zi+\tfrac{1}{2}-\tfrac{1}{2}h\right)\right\} = \frac{1}{z-}\,\frac{c_1}{z-}\,\frac{c_2}{z-}\cdots.$$

The denominators of the convergents of (3.1) are orthogonal polynomials with respect to the distribution function $\Psi(x)$ obtained by inverting the STIELTJES transform

$$\frac{i}{2h}\left\{\Psi\left(\tfrac{1}{2}zi+\tfrac{1}{2}+\tfrac{1}{2}h\right)-\Psi\left(\tfrac{1}{2}zi+\tfrac{1}{2}-\tfrac{1}{2}h\right)\right\} = \int_{-\infty}^{\infty}\frac{d\,\Psi(u)}{z-u}.$$

In fact we know that [7, p. 253]

$$\Psi(u) = \lim_{y=0}\int_0^u \operatorname{Im}\frac{i}{h}\left\{\Psi\left(ix-y+\tfrac{1}{2}-\tfrac{1}{2}h\right)-\Psi\left(ix-y+\tfrac{1}{2}-\tfrac{1}{2}h\right)\right\}dx.$$

If $p(u)$ exists such that $\Psi'(u) = p(u)$ then

$$p(u) = \lim_{y=+0}\operatorname{Im}\frac{i}{h}\left\{\Psi\left(ix-y+\tfrac{1}{2}+\tfrac{1}{2}h\right)-\Psi\left(ix-y+\tfrac{1}{2}-\tfrac{1}{2}h\right)\right\}.$$

To calculate $p(x)$ we make use of the formula [4, p. 102]

$$\Psi(z+1) = -\,C + \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$$

where C is EULER's constant. We get

$$p(x) = \frac{1}{\pi h}\sum_{n=1}^{\infty}\left\{\frac{2n-1-h}{(2n-h-1)^2+x^2}-\frac{2n-1+h}{(2n+h-1)^2+x^2}\right\}$$

$$= \frac{\sin \pi h}{2h}\,\frac{1}{\cos\frac{\pi}{2}(h-ix)\,\cos\frac{\pi}{2}(h+ix)}.$$

Replace ix by $2x - h + 1$ we get the weight function associated with the polynomials $\phi_n(x)$, i.e.,

$$p(x) = \frac{\sin \pi h}{2h} \frac{1}{\sin \pi (x - h) \sin \pi x}.$$

From (1.2) we have

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} \frac{C_n}{2^n} \frac{t^n}{n!}.$$

We thus have asymptotic formula

$$2 \int_0^{\infty} \frac{e^{-xt}}{e^t + 1} dt = 2 \int_0^{\infty} \operatorname{sech} t \, e^{-t(2x+1)} \, dt \simeq \sum_{n=0}^{\infty} \frac{2^{-n} C_n}{x^{n+1}}.$$

If we recall the continued fraction expansion [7, p. 206]

$$\text{(3.2)} \qquad \int_0^{\infty} \operatorname{sech}^k t \, e^{-zt} \, dt = \frac{1}{z+} \frac{1.k}{z+} \frac{2.(k+1)}{z+} \frac{3.(k+2)}{z+} \cdots$$

it follows that $2^{-n} C_n$ are the moments associated with the orthogonal polynomials defined by means of

$$f_{n+1}(x) = (2x + 1) f_n(x) + n^2 f_{n-1}(x)$$

$$f_0(x) = 1, \ f_1(x) = x + \tfrac{1}{2}.$$

But $f_n(x) = n! \, A_n(x)$. We thus have (1.8).

In the same way as in the proof of (1.7) we can obtain the actual orthogonality satisfied by the polynomials $A_n(z)$. This has been essentially obtained by different methods in [1] and [3].

REFERENCES

1. L. WEISNER. — *Problem n.º 4766 American Mathematical Monthly*, vol. 65 (1958), p. 714.

2. L. CARLITZ. — *Bernoulli and Euler numbers and orthogonal polynomials*. Duke Mathematical Journal, vol. 26 (1959), pp. 1-16.

3. G. H. HARDY. — *Notes on special systems of orthogonal functions* (III) : *A system of orthogonal polynomials*. Proc. of the Cambridge Philosophical Society, vol. 36 (1940), pp. 1 8.

4. N. NORLUND. — *Vorlesungen uber Differenzenrechnung*. Berlin, 1924.

5. S. PASTERNACK. — *A generalization of the polynomials* $F_n(z)$. Phil. Mag. (7), vol. 28 (1939), pp. 209 226.

6. J. TOUCHARD. — *Nombres exponentiels et nombres de Bernoulli*. Canadian Journal of Mathematics, vol. 8 (1956), pp. 305-320.

7. H. S. WALL. — *Analytic Theory of Continued Fractions*, New York, 1948.