INTEGRALS INVOLVING G-FUNCTION

by

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1. The object of this paper is to evaluate some integrals involving MEIJER's G-function.

2. We require the following results in the investigation.

(*a*) The Gamma- function formula [(1) p. 4]

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}-\frac{1}{2}m}\,(m)^{mz-\frac{1}{2}} \prod_{k=0}^{m-1} \Gamma\left(z+\frac{k}{m}\right)$$

where m is a positive integer. (1)

(*b*) The definition of MEIJER's G-function [(1) p. 207]

$$G^{m,\,n}_{p,\,q}\left(x \,\middle|\, \begin{matrix}\alpha_1 \ldots \alpha_p \\ \beta_1 \ldots \beta_q\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(\beta_j - s)\prod_{j=1}^{n}\Gamma(1-\alpha_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j+s)\prod_{j=n+1}^{p}\Gamma(\alpha_j - s)}\, x^s\, ds$$

where L is a suitable contour. (2)

(*c*) The infinite integral

$$\int_0^{\infty} e^{2\mu\theta}(\sinh\theta)^{2\lambda-1}\,{}_2F_1\left(\lambda-\mu+\tfrac{1}{2},\,\beta;\,\delta;\,2\,e^{-\theta}\sinh\theta\right)d\theta$$

$$= \frac{\Gamma(\delta)\Gamma(\lambda)\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\frac{1}{4}-\frac{\lambda}{2}-\frac{\mu}{2}\right)\Gamma\left(\frac{3}{4}-\frac{\lambda}{2}-\frac{\mu}{2}\right)\Gamma\left(\frac{\delta}{2}-\frac{\beta}{2}-\lambda\right)\Gamma\left(\frac{1}{2}+\frac{\delta}{2}-\frac{\beta}{2}-\lambda\right)}{\pi\, 2^{\lambda+\mu+\beta+\frac{3}{2}}\,\Gamma(\delta-\beta)\,\Gamma\left(\frac{1}{2}+\frac{\lambda}{2}-\frac{\mu}{2}\right)\Gamma\left(\frac{\delta}{2}-\lambda\right)\Gamma\left(\frac{1}{2}+\frac{\delta}{2}-\lambda\right)}.$$

valid for $R(\lambda)>0$, $R(1/2-\lambda-\mu)>0$, $R(\delta-\beta-\lambda+\mu-1/2)>0$. (3)

(3) Can be established by writing down the series for the hy-

pergeometric function in the integrand and integrating term by term with the help of the result [(2) p. 311].

$$\int_0^\infty e^{2\mu\theta} (\sinh \theta)^{2\lambda-1} \, d\theta = \frac{2^{-2\lambda} \Gamma(2\lambda) \Gamma\left(\frac{1}{2} - \lambda - \mu\right)}{\Gamma\left(\frac{1}{2} + \lambda - \mu\right)}$$

valid for $R(\lambda) > 0$, $R(\frac{1}{2} - \lambda - \mu) > 0$, (4)

(*d*) The definite integrals

$$\int_0^{\pi/2} e^{i(\alpha+\beta)\theta} (\sin \theta)^{\alpha-1} (\cos \theta)^{\beta-1} {}_2F_1(a, b; \beta; e^{i\theta} \cos \theta) \, d\theta$$

$$= e^{i\frac{\pi}{2}\alpha} \frac{\Gamma(\alpha) \Gamma(\beta) \Gamma(\alpha+\beta-a-b)}{\Gamma(\alpha+\beta-a) \Gamma(\alpha+\beta-b)}$$

valid for $R(\alpha) > 0$, $R(\beta) > 0$, $R(\beta - a - b) > 0$, (5)

(*e*) $$\int_0^{\pi/2} e^{i(\alpha+\beta)\theta} (\sin \theta)^{\alpha-1} (\cos \theta)^{\beta-1} {}_2F_1[a, b; \alpha; e^{i(\theta-\frac{\pi}{2})} \sin \theta] \, d\theta$$

$$= e^{i\frac{\pi}{2}\alpha} \frac{\Gamma(\alpha) \Gamma(\beta) \Gamma(\alpha+\beta-a-b)}{\Gamma(\alpha+\beta-a) \Gamma(\alpha+\beta-b)}.$$

valid for $R(\alpha) > 0$, $R(\beta) > 0$, $R(\alpha - a - b) > 0$, (6)

(5) and (6) can be proved in a similar manner by using the integral [(5). p. 450].

$$\int_0^{\pi/2} e^{i(\alpha+\beta)\theta} (\sin \theta)^{\alpha-1} (\cos \theta)^{\beta-1} \, d\theta = e^{i\frac{\pi}{2}\alpha} \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

valid for $R(\alpha) > 0$, $R(\beta) > 0$, (7)

(*f*) The Infinite Integrals

$$\int_0^\infty x^{\lambda-1} (1+x)^{\lambda-a-1} {}_4F_3\left[\begin{matrix} a, \frac{1}{2}a+1, c. d; \frac{x}{1+x} \\ \frac{1}{2}a, 1+a-c. 1+a-d \end{matrix}\right] dx$$

$$= \frac{2^{a-2\lambda} \Gamma(1+a-c) \Gamma(1+a-d) \Gamma(\frac{1}{2}+\frac{1}{2}a-\lambda) \Gamma(1+\frac{1}{2}a-\lambda) \Gamma(1+a-c-d-\lambda)}{\sqrt{\pi}\, \Gamma(1+a) \Gamma(1+a-c-d) \Gamma(1+a-d-\lambda) \Gamma(1+a-c-\lambda) [\Gamma(\lambda)]^{-1}}$$

valid for $R(\lambda) > 0$, $R(a-2\lambda+1) > 0$. $R(a-c-d-\lambda+1) > 0$ (8)

and

$$\int_0^\infty x^{\lambda-1} (1+x)^{\lambda-a-1} {}_3F_2 (a, \tfrac{1}{2} a+1, b; \tfrac{1}{2} a, 1+a-b; -\frac{x}{1+x}) dx$$

$$= \frac{2^{a-2\lambda} \Gamma(1+a-b) \Gamma(\lambda) \Gamma(\frac{1}{2}+\frac{1}{2} a-\lambda) \Gamma(1+\frac{1}{2} a-\lambda)}{\sqrt{\pi} \Gamma(1+a) \Gamma(1+a-b-\lambda)}$$

valid for $R(\lambda) > 0$, $R(a-2\lambda+1) > 0$, $R(a-2b-2\lambda+2) > 0$. (9)

These results have been derived with the help of

$$\int_0^\infty x^{\alpha-1} (1+x)^{-\alpha-\beta} dx = B(\alpha, \beta).$$

valid for, $R(\alpha) > 0$, $R(\beta) > 0$,
DOUGALL's theorem [(6), p. 372] and whipples' theorem [(6) p. 368].

3. THE FIRST INTEGRAL : — The formula to be proved is

$$\int_0^\infty e^{2\mu\theta} (\sinh \theta)^{2\lambda-1} {}_2F_1 (\tfrac{1}{2}+\lambda-\mu, \beta; \delta; 2 e^{-\theta} \sinh \theta)$$

$$G^{p,\, q}_{r,\, s} \left[e^{2m\theta} (\sinh \theta)^{2m} z \,\middle|\, \begin{matrix} \alpha_1 \ldots\ldots \alpha^r \\ \beta_1 \ldots\ldots \beta_s \end{matrix} \right] d\theta$$

$$= \frac{(\pi)^{1-2m} (m)^{\lambda-\mu-\frac{\beta}{2}-\frac{1}{2}} \Gamma(\delta)}{2^{\lambda+\mu+\beta-\frac{1}{2}} \Gamma(\delta-\beta) \Gamma\left(\frac{1}{2}+\lambda-\mu\right)}$$

$$G^{p+4m,\, q+2m}_{r+4m,\, s+4m} \left[\frac{z}{2^{2m}} \,\middle|\, \begin{matrix} \varphi_1 \ldots \varphi_{2m}, \alpha_r, \ldots \alpha, \varphi_{2m+1} \ldots \varphi_{4m} \\ \psi_1 \ldots \psi_{4m}, \beta_1, \ldots \ldots \beta_s \end{matrix} \right]$$

where m is a positive integer, and (10)

$$\varphi_{k+1} = \frac{1-\lambda+k}{m}, \quad \varphi_{k+m+1} = \frac{1-2\lambda+2k}{2m},$$

$$\varphi_{k+2m+1} = \frac{1+\delta-2\lambda+2k}{2m}, \quad \varphi_{k+3m+1} = \frac{\delta-2\lambda+2k}{2m}$$

$$\psi_{k+1} = \frac{1-2\lambda-2\mu+4k,}{4m}, \quad \psi_{k+m+1} = \frac{3-2\lambda-2\mu+4k}{4m}$$

$$\psi_{k+2m+1} = \frac{\delta-\beta-2\lambda+2k}{2m}, \quad \psi_{k+3m+1} = \frac{1-\beta+\delta-2\lambda+2k}{2m},$$

$$k = 0,\ 1,\ 2,\ \ldots\ldots\ (m-1)$$

The result is valid under the following conditions

(1) $r + s < 2\ (p+q),\ |\arg z| < (p+q-\frac{1}{2}\,r-\frac{1}{2}\,s)\,\pi,$

$R\,(\delta-\beta-\lambda+\mu-\frac{1}{2}) > 0,\ R\,(\lambda+m\beta_j) > 0,$ for $j = 1, 2, \ldots p.$

$R\,(\lambda+\mu-2m\ -2m\alpha_h-\frac{1}{2}) < 0$ for $h = 1,\ 2,\ \ldots\ q.$

(*ii*) $r \leq s,\ r+s \leq 2\ (p+q),\ |\arg z| \leq (p+q-\frac{1}{2}\,r-\frac{1}{2}\,s)\,\pi,$

$R\,(\delta-\beta-\lambda+\mu-\frac{1}{2}) > 0,\ R\,(\lambda+m\beta_j)\ 0,$ for $j = 1,\ 2,\ \ldots\ p.$

$$R\,(\lambda+\mu-2m\alpha_h) < \frac{4m+1}{2},\ \text{for } h = 1, 2, \ldots q.$$

$$2m\left(\sum_{j=1}^{r}\alpha_j - \sum_{j=1}^{s}\beta_j.-\tfrac{1}{2}\right) - (s-r)\,(\lambda+\mu-m-\tfrac{1}{2}) > 0$$

(*iii*) $r \geq s,\ r+s \leq 2\,(p+q),\ |\arg z| \leq (p+q-\frac{1}{2}\,r-\frac{1}{2}\,s)\,\pi,$

$R\,(\delta-\beta-\lambda+\mu-\frac{1}{2}) > 0,\ R\,(\lambda+m\beta_j) > 0$ for $j = 1,\ 2,\ \ldots\ p.$

$$R\,(\lambda+\mu-2\ m\ \alpha_h) < \frac{4m+1}{2},\ \text{for } h = 1,\ 2,\ \ldots\ q.$$

$$2m\left(\sum_{j=1}^{r}\alpha_j - \sum_{j=1}^{s}\beta_j+\tfrac{1}{2}\right) + (r-s)\,(\lambda-m) > 0.$$

To prove (10), we substitute the contour integral (2) for the Meijer's G-function in the integrand of (10), change the order of integration (which we suppose to be permissible) and evaluate the inner integral with the help of (3).

The value of the integral then becomes

$$\frac{(2)^{-\lambda-\mu-\beta-\frac{3}{2}}\ \Gamma(\delta)}{\Gamma(\delta-\beta)\,\Gamma\left(\frac{1}{2}+\lambda-\mu\right)\pi}\ \frac{1}{2\pi i}\int \frac{\prod_{j=1}^{p}\Gamma(\beta_j-\xi)\prod_{j=1}^{q}\Gamma(1-\alpha_j+\xi)}{\prod_{j=p+1}^{s}\Gamma(1-\beta_j+\xi)\prod_{j=q+1}^{r}\Gamma(\alpha_j-\xi)}$$

$$\frac{\Gamma(\lambda+m\xi)\,\Gamma\left(\frac{1}{2}+\lambda+m\xi\right)\Gamma\left(\frac{1}{4}-\frac{\lambda}{2}-\frac{\mu}{2}-m\xi\right)\Gamma\left(\frac{3}{4}-\frac{\lambda}{2}-\frac{\mu}{2}-m\xi\right)}{\Gamma\left(\frac{\delta}{2}-\lambda-m\xi\right).}\times$$

$$\frac{\Gamma\left(\frac{\delta}{2}-\frac{\beta}{2}-\lambda-m\xi\right)\Gamma\left(\frac{1}{2}+\frac{\delta}{2}-\beta/2-\lambda-m\xi\right).\ [z2^{-2m}z]^{\xi}\,d\xi.}{\Gamma\left(\frac{1}{2}+\frac{\delta}{2}-\lambda-m\xi\right).}=$$

$$=\frac{(m)^{\lambda-\mu-\frac{\beta}{2}-\frac{1}{2}}(\pi)^{1-2m}\,\Gamma(\delta)}{2^{\lambda+\mu+\beta-\frac{1}{2}}\Gamma(\delta-\beta)\,\Gamma\left(\frac{1}{2}+\lambda-\mu\right)}\ \frac{1}{2\pi i}\int\frac{\prod_{j=1}^{p}\Gamma(\beta_j-\xi)\prod_{j=1}^{q}\Gamma(1-\alpha_j+\xi).}{\prod_{j=p+1}^{s}\Gamma(1-\beta_j+\xi)\prod_{j=q+1}^{r}\Gamma(\alpha_j-\xi).}\times$$

$$\frac{\prod_{k=0}^{m-1}\Gamma\left(\frac{k+\lambda}{m}+\xi\right)\prod_{k=0}^{m-1}\Gamma\left(\frac{1+2\lambda+2k}{2m}+\xi\right)\prod_{k=0}^{m-1}\Gamma\left(\frac{1-2\lambda-2\mu+4k}{4m}-\xi\right).}{\prod_{k=0}^{m-1}\Gamma\left(\frac{1+\delta-2\lambda+2k}{2m}-\xi\right).}\times$$

$$\frac{\prod_{k=0}^{m-1}\Gamma\left(\frac{3-2\lambda-2\mu+4k}{4m}-\xi\right)\prod_{k=0}^{m-1}\Gamma\left(\frac{\delta-\beta-2\lambda+2k}{2m}-\xi\right)\prod_{k=0}^{m-1}\Gamma\left(\frac{1+\delta-2\lambda-\beta+2k}{2m}-\xi\right)}{\prod_{k=0}^{m-1}\Gamma\left(\frac{\delta-2\lambda+2k}{2m}-\xi\right)}\times$$

$$\left[\frac{z}{2^{2m}}\right]^{\xi}d\xi. \tag{11}$$

by virtue of (1).

The contour L runs from- $i\infty$ to $+i\infty$, so that all the poles of $\Gamma(\beta_j-\xi)$, for $j=1, 2\ldots p$, $\Gamma\left(\frac{1-2\lambda-2\mu+4k}{4m}-\xi\right)$, $\Gamma\left(\frac{3-2\lambda-2\mu+4k}{4m}-\xi\right)$, $\Gamma\left(\frac{\delta-\beta-2\lambda+2k}{2m}-\xi\right)$ and $\Gamma\left(\frac{1+\delta-2\lambda-\beta+2k}{2m}-\xi\right)$, for $k=0, 1, 2, \ldots (m-1)$ are to the right, and all the poles of $\Gamma(1-\alpha_j+\xi)$ for $j=1, 2, \dots q$,

$$\Gamma\left(\frac{k+\lambda}{m}+\xi\right) \text{ and } \Gamma\left(\frac{1+2k+2\lambda}{2m}+\xi\right), \text{ for } k=0, 1, \dots m\text{-}1,$$

are to the left of L.

Interpreting (11), with the help of (2), we get (10) under the conditions stated there in.

In particular, on taking $\beta=0$ and making the substitution $x=\sinh^2\theta$ in (10), it yields the known result [(4) p. 347].

The following integrals (12), (13), (14) and (15) can be derived in the same way from the formulae (6), (7), (8) and (9) respectively.

4. The second integral: —

$$\int_0^{\pi/2} e^{i(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}\, {}_2F_1\left[a, b; \alpha; e^{-i\left(\frac{\pi}{2}-\theta\right)}\sin\theta\right]$$

$$G^{p,\,q}_{r,\,s}\left[e^{-im\theta}(\cos\theta)^m z\,\middle|\,\begin{matrix}\alpha_1\dots\alpha_r\\ \beta_1\dots\beta_s\end{matrix}\right]d\theta$$

$$=\frac{e^{i\frac{\pi}{2}\alpha}}{m^\alpha}\Gamma(\alpha)\, G^{p,\,q+2m}_{r+2m,\,s+2m}\left[z\,\middle|\,\begin{matrix}\varphi_1\dots\varphi_{2m}, \alpha_1\dots\alpha_r\\ \beta_1\dots\beta_s, \psi_1\dots\psi_{2m}\end{matrix}\right]$$

where m is a positive integer, and (12)

$$\varphi_{k+1}=\frac{1-\beta+k}{2m},\quad \varphi_{k+m+1}=\frac{1+a+b-\alpha-\beta+k}{m},$$

$$\psi_{k+1}=\frac{1+a-\alpha-\beta+k}{m},\quad \psi_{k+m+1}=\frac{1+b-\alpha-\beta+k}{m},$$

$k=0, 1, \dots (m-1)$. The result is valid under the following conditions.

(*i*) $0\le p\le s,\ 0\le q\le r<s$ (or $r=s$ and $|z|<|$)

$R(\alpha)>0,\ R(\alpha-a-b)>0,\ R(\beta+m\,\beta_j)>0,\ (j=1, 2 \dots\dots p)$.

(*ii*) $0\le q\le r,\ 1\le p\le s<r,\ p+q>\frac{1}{2}(r+s),\ |\arg z|<(p+q-\frac{1}{2}r-\frac{1}{2}s)\pi$,

$R(\alpha)>0,\ R(\alpha-a-b)>0,\ R(\beta+m\,\beta_j)>0,\ j=1, 2 \dots\dots p$.

(*iii*) $r \geq s,\ p + q \geq \frac{1}{2}(r + s),\ |\arg z| \leq (p + q - \frac{1}{2}r - \frac{1}{2}s)\pi,$

$R(\alpha) > 0,\ R(\alpha - a - b) > 0,\ R(\beta + m\ \beta_j) > 0,\ j = 1, 2 \ldots\ldots p,$

$$m\left[\sum_{j=1}^{r} \alpha_j - \sum_{j=1}^{s} \beta_j + \frac{1}{2}\right] + (r - s)\left(\beta - \frac{m}{2} - 1\right) > 0.$$

5. The third integral : —

$$\int_0^{\pi/2} e^{i(\alpha+\beta)\theta} (\sin\theta)^{\alpha-1} (\cos\theta)^{\beta-1} {}_2F_1(a, b; \beta; e^{i\theta}\cos\theta)$$

$$\times G_{r,\ s}^{p,\ q}\left[e^{im(\theta-\frac{\pi}{2})} (\sin\theta)^m z \,\middle|\, \begin{matrix}\alpha_1 \ldots \alpha_r\\ \beta_1 \ldots \beta_s\end{matrix}\right] d\theta$$

$$= \frac{e^{i\frac{\pi}{2}\alpha}\, \Gamma(\beta)}{m^{\beta}} \times$$

$$G_{r+2m,\ s+2m}^{p,\ q+2m}\left[z \,\middle|\, \begin{matrix}\varphi_1 \ldots \varphi_{2m},\ \alpha_1 \ldots \alpha_r\\ \beta_1 \ldots \beta_s,\ \psi_1 \ldots \psi_{2m}\end{matrix}\right]$$

where m is a positive integer and (13)

$$\varphi_{k+1} = \frac{1 - \alpha + k}{m},\ \varphi_{K+m+1} = \frac{1 + a + b - \alpha - \beta + k}{m}$$

$$\psi_{k+1} = \frac{1 + a - \alpha - \beta + k}{m},\ \psi_{k+m+1} = \frac{1 + b - \alpha - \beta + k}{m}$$

$k = 0,\ 1,\ \ldots\ (m - 1).$

(13) is valid under the conditions stated with (12) if we interchange the place of α and β.

6. The fourth integral : —

$$\int_0^{\infty} x^{\lambda-1} (1 + x)^{\lambda-a-1} {}_4F_3\left[\begin{matrix}a,\ \frac{1}{2}a + 1,\ c,\ d,\ \dfrac{x}{1+x}\\ \frac{1}{2}a,\ 1 + a - c,\ 1 + a - d\end{matrix}\right]$$

$$\times G_{r,\ s}^{p,\ q}\left[z(4x^2 + 4x)^n \,\middle|\, \begin{matrix}\alpha_1 \ldots \alpha_r\\ \beta_1 \ldots \beta_s\end{matrix}\right] dx$$

$$= \frac{2^{1+a-2\lambda-n}\, \Gamma(1 + a - c)\, \Gamma(1 + a - d)}{\sqrt{n}\, (\pi)^{n-\frac{1}{2}}\, \Gamma(1 + a)\, \Gamma(1 + a - c - d)}$$

$$G^{p+3n,\ q+n.}_{r+3n,\ s+3n}\left[z \;\middle|\; \begin{matrix} \varphi_1 \ldots \varphi_n, \ \alpha_1 \ldots \alpha_r, \ \varphi_{n+1} \ldots \varphi_{3n} \\ \psi_1, \ldots \psi_{3n}, \ \beta_1 \ldots \ldots \beta_s \end{matrix} \right]$$

where n is a positive integer, and (14)

$$\varphi_{k+1} = \frac{1-\lambda+k}{n}, \quad \varphi_{k+n+1} = \frac{1+a-d-\lambda+k}{n},$$

$$\varphi_{k+2n+1} = \frac{1+a-c-\lambda+k}{n}, \quad \psi_{k+1} = \frac{1+a-2\lambda+2k}{2n}$$

$$\psi_{k+n+1} = \frac{2+a-2\lambda+2k}{2n}, \quad \psi_{k+2n+1} = \frac{1+a-c-d-\lambda+k}{n}$$

$k = 0, 1, \ldots (n-1)$, the result is valid under the following conditions

(*i*) $(r+s) < 2(p+q)$, $|\arg z| < (p+q-\frac{1}{2}r-\frac{1}{2}s)\,\pi$,

$R(a-c-d-\lambda+1) > 0$, $R(\lambda+2n\ \beta_j) > 0$, $j = 1, 2, \ldots p$,

$R(2\lambda - a + 2n\alpha_h - 2n - 1) < 0$, $h = 1, 2, \ldots q$.

(*ii*) $r \leq s$, $(r+s) \leq 2(p+q)$, $|\arg z| \leq (p+q-\frac{1}{2}r-\frac{1}{2}s)\,\pi$.

$R(a-c-d-\lambda+1) > 0$, $R(\lambda+2n\ \beta_j) > 0$, $j = 1, 2, \ldots p$.

$R(2\lambda - a + 2n\alpha_h - 2n - 1) < 0$, $h = 1, 2, \ldots q$.

$$2n\left[\sum_{j=1}^{r} \alpha_j - \sum_{j=1}^{s} \beta_j - \frac{1}{2}\right] - (s-r)\,(2\lambda - n - a - 1) > 0.$$

(*iii*) $r \geq s$, $(r+s) \leq 2(p+q)$, $|\arg z| \leq (p+q-\frac{1}{2}r-\frac{1}{2}s)\,\pi$,

$R(a-c-d-\lambda+1) > 0$, $R(\lambda+2n\ \beta_j) > 0$, $j = 1, 2, \ldots p$.

$R(2\lambda - a + 2n\,\alpha_h - 2n - 1) < 0$, $h = 1, 2, \ldots q$.

$$2n\left[\sum_{j=1}^{r} \alpha_j - \sum_{j=1}^{s} \beta_j + \frac{1}{2}\right] + (r-s)\,(\lambda - n) > 0.$$

7. The fifth integral: —

$$\int_0^\infty x^{\lambda-1}\,(1+x)^{\lambda-a-1}\ {}_3F_2\left(a, \tfrac{1}{2}a+1, b;\ \tfrac{1}{2}a, 1+a-b;\ -\frac{x}{1+x}\right) \times$$

$$G_{r,\ s}^{p,\ q}\left[z\,(4x^2+x)^n \,\middle|\, \begin{matrix}\alpha_1 \ldots \alpha_r\\ \beta_1 \ldots \beta_s\end{matrix}\right] dx$$

$$= \frac{(2)^{a-n-2\lambda-1}\ (n)^{b-\frac{1}{2}}\ \Gamma(1+a-b)}{(\pi)^{n-\frac{1}{2}}\ \Gamma(1+a)}$$

$$G_{r+2n,\ s+2n}^{p+2n,\ q+n.}\left[Z \,\middle|\, \begin{matrix}\varphi_1 \ldots \varphi_n,\ \alpha_1 \ldots \alpha_r,\ \varphi_{n+1}, \ldots \varphi_{2n}\\ \psi_1 \ldots \psi_{2n},\ \beta_1 \ldots \qquad \ldots \beta_s\end{matrix}\right]$$

where n is a positive integer (15)

$$\varphi_{k+1} = \frac{1-\lambda+k}{n},\ \varphi_{k+n+1} = \frac{1+a-\lambda-b+k}{n},$$

$$\psi_{k+1} = \frac{1+a-2\lambda+2k}{2n},\ \psi_{k+n+1} = \frac{2+a-2\lambda+2k}{2n}$$

$k = 0, 1, \ldots, n-1$. The result is valid under the conditions, stated with (14) and having

$R\,(a-2\,b-2\,\lambda+2) > 0$, instead of $R\,(a-c-d-\lambda+1) > 0$.

A few interesting particular cases of the general result (15) are given below :

(*i*) Assuming $n = 1$, $s = p = 2$, $q = r = 0$, $\beta_1 = v$, $\beta_2 = -v$ and using the relation [(1), 216.]

$$G_{0\ 2}^{2\ 0}\left(x \,\middle|\, a,\ b\right) = 2x^{\frac{1}{2}(a+b)}\ K_{a-b}\ (2\sqrt{x})$$

in (15), we have the following integral

$$\int_0^{\infty} x^{\lambda-1}\ (1+x)^{\lambda-a-1}\ {}_3F_2\left[\begin{matrix}a,\ \frac{1}{2}\,a+1,\ b\ ;\ -\dfrac{x}{1+x}\\ \dfrac{1}{2}\,a,\ 1+a-b\end{matrix}\right] k_{2v}\left[4\sqrt{z\,(x^2+x)}\right] dx$$

$$= \frac{2^{a-2\lambda-1}\ \Gamma(1+a-b)}{\sqrt{\pi}\ \Gamma(1+a)}\ z^{-\lambda}\ E\left(\frac{1}{2}+\frac{a}{2},\ \frac{a}{2}+1,\ v+\lambda,\ \lambda-v\ ;\ 1+a-b\ ;\ z\right)$$

provided $R(\lambda \pm v) > 0$, $|\arg z| < \pi$. (16)

further taking $\lambda = \frac{1}{2} - k$, $b = 0$, $a = -2k$, $v = m$ and $z = p/4$, we obtain a known result [(3), p. 377].

(*ii*) Assuming $n = 1$, $p = 2$, $q = 0$, $r = 1$, $s = 2$, $\alpha_1 = 1 - k$, $\beta_1 = \frac{1}{2} + m$, $\beta_2 = \frac{1}{2} - m$, $a = 1$, $b = \frac{1}{2}$, $\lambda = k$ in (15) and using the relation [(1), p. 217]

$$G_{12}^{20}\left(x \,\middle|\, \begin{matrix} 1-k \\ \frac{1}{2}+m, \frac{1}{2}-m \end{matrix} \right) = e^{-\frac{1}{2}x} W_{k,m}(x)$$

we have a new integral representation for WHITTAKER function,

$$\int_0^\infty x^{k-1}(1+x)^{k+1}(1+2x)^{-1} e^{-2z(x^2+x)} W_{k,m}[4z(x^2+x)]\,dx$$

$$= \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+k \pm m\right)}{2^{2k}} e^{\frac{1}{2}z} W_{-k,m}(z).$$

valid for $R(\frac{1}{2} + k \pm m) > 0$, $|\arg z| < \frac{\pi}{2}$. (17)

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