SOME FORMULAE FOR THE G-FUNCTION-II

by

R. K. SAXENA

1. INTRODUCTION. In a recent paper [7] the author has evaluated certain integrals involving MEIJER'S G-functions in which the argument of the G-function contains a factor $t^m \{ t^{\frac{1}{2}} + (1+t)^{\frac{1}{2}} \}^{2n}$, where m and n are positive integers and t is the variable of integration. In the present note five integrals involving G-functions of different arguments will be evaluated with the help of certain Lemmas involving the LAPLACE transformation established in § 2. The results proved here are of general character and several results given earlier by MACROBERT [**2**], [**3**] follow as particular cases of our formulae. For the definition, properties and the behaviour of the G-function see [**2**, §§ 5.3 and 5.3.1] and [**5**, § 18].

The conventional notation $\Phi(p) \doteqdot h(t)$ will be used to denote the classical LAPLACE'S integral

$$(1.1) \qquad \Phi(p) = p \int_0^\infty e^{-pt} \, h(t) \, dt.$$

In what follows m and n are positive integers and the symbol $\Delta(n\,;\,\alpha)$ denotes the set of parameters

$$\frac{\alpha}{n}, \frac{\alpha+1}{n}, \ldots, \frac{\alpha+n-1}{n}.$$

The following results will be required in the sequel.

If $R(\nu) > 0$, $R\left(\nu - \frac{1}{2}\alpha - \frac{1}{2}\beta + \frac{1}{2}\right) > 0$, then

$$(1.2) \qquad \int_0^1 t^{\nu-1} (1-t)^{\nu-1} \, {}_2F_1\left(\alpha, \beta\,;\, \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2}\,;\, t\right) \times \exp\{-x t^m (1-t)^m\} \, dt$$

$$= \frac{\pi m^{-1/2}\, 2^{1-2\nu}\, \Gamma\left(\frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\alpha + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta + \frac{1}{2}\right)}$$

$$\times E\left[\Delta(m;\ \nu), \Delta\left(m; \nu - \frac{1}{2}\alpha - \frac{1}{2}\beta + \frac{1}{2}\right) : \Delta\left(m; \nu - \frac{1}{2}\alpha + \frac{1}{2}\right),\right.$$

$$\left.\Delta\left(m; \nu - \frac{1}{2}\beta + \frac{1}{2}\right) :\ 2^{2m}/x\right]$$

If $R(\beta) > 0, R(\beta - \nu + 1) > 0,$ then

$$(1.3) \quad \int_0^1 t^{\beta-1}(1-t)^{\beta-\nu}\, {}_2F_1(\alpha,\ 1-\alpha;\ \nu;\ t)\exp\{-xt^m(1-t)^m\}\,\mathrm{dt}$$

$$= \frac{\pi\, m^{-1/2}\, 2^{1-2\beta}\, \Gamma(\nu)}{\Gamma\left(\frac{1}{2}\alpha + \frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}\alpha + \frac{1}{2}\nu\right)}$$

$$\times E\left[\Delta(m;\beta), \Delta(m;\beta - \nu + 1) : \Delta\left(m;\beta + \frac{1}{2}\alpha - \frac{1}{2}\nu + \frac{1}{2}\right),\right.$$

$$\left.\Delta\left(m;\beta - \frac{1}{2}\alpha - \frac{1}{2}\nu\right) : \frac{2^{2m}}{x}\right]$$

If $R(\alpha) > 0,\ R(\beta) > 0,$ then

$$(1.4) \quad \int_0^{\pi/2} e^{i(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}\exp\{-x\, e^{im\theta}\sin^m\theta\}\,d\theta$$

$$= e^{i\frac{1}{2}\pi\alpha}\, m^{-\beta}\, \Gamma(\beta)$$

$$\times E\left[\Delta(m;\alpha) : \Delta(m;\alpha + \beta) : \frac{e^{-i\frac{1}{2}m\pi}}{x}\right],$$

$$(1.5) \quad \int_0^{\pi/2}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}\exp\{i(\alpha+\beta)\theta - xe^{i(m+n)\theta}\sin^m\theta\cos^n\theta\}\,d\theta$$

$$= e^{i\frac{1}{2}\pi\alpha}\sqrt{2\pi}\,\frac{m^{\alpha-1/2}\, n^{\beta-1/2}}{(m+n)^{\alpha+\beta-1/2}}$$

$$\times E\left[\begin{matrix}\Delta(m;\ \alpha),\ \Delta(n;\ \beta)\\ \Delta(m+n;\alpha+\beta)\end{matrix} : \frac{e^{-i\frac{1}{2}\pi m}(m+n)^{m+n}}{m^m n^n x}\right],$$

(1.6) $$\int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{\{1+ct+d(1-t)\}^{\alpha+\beta}} \exp\left[-\frac{xt^m(1-t)^n}{\{1+ct+d(1-t)\}^{m+n}}\right] dt$$

$$= \frac{\sqrt{2\pi}\ m^{\alpha-1/2}\ n^{\beta-1/2}}{(1+c)^\alpha (1+d)^\beta (m+n)^{\alpha+\beta-1/2}}$$

$$\times E\left[\begin{matrix}\Delta(m;\ \alpha),\ \Delta(n;\ \beta)\\ \Delta(m+n;\alpha+\beta)\end{matrix} : \frac{(m+n)^{m+n}(1+c)^m(1+d)^n}{x\ m^m\ n^n}\right],$$

where c and d are such that no one of the expression $1+c$, $1+d$, $1+ct+d(1-t)$, where $0 \leq t \leq 1$ is zero.

(1.2), (1.3), (1.4), (1.5) and (1.6) follow from the results given by MacRobert [**2**] and [**3**].

2. Lemma 1. If

$$\Phi(p) \doteqdot h(x),$$

then

(2.1) $$\int_0^1 t^{\nu-m-1}(1-t)^{\nu-m-1}\,{}_2F_1\left(\alpha,\ \beta;\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2};t\right)$$

$$\times\ \Phi\{t^m(1-t)^m\}\ dt$$

$$= \frac{\pi\ m^{-1/2}\ 2^{1-2\nu}\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}\int_0^\infty h(x)$$

$$\times E\left[\Delta(m;\nu),\ \Delta\left(m;\nu-\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{1}{2}\right):\ \Delta\left(m;\ \nu-\frac{1}{2}\alpha+\frac{1}{2}\right),\right.$$

$$\left.\Delta\left(m;\ \nu-\frac{1}{2}\beta+\frac{1}{2}\right):\frac{2^{2m}}{x}\right]dx,$$

provided the integrals are absolutely convergent, $R(\nu) > 0$ and $R\left(\nu-\frac{1}{2}\alpha-\frac{1}{2}\beta\right) > -1/2$;

Lemma 2. If

$$\Phi(p) \doteqdot h(x),$$

then

$$\text{(2.2)} \quad \int_0^1 t^{\beta-m-1}(1-t)^{\beta-m-\nu}\, {}_2F_1(\alpha, 1-\alpha; \nu; t) \times \Phi\{t^m(1-t)^m\}\, dt$$

$$= \frac{\pi\, m^{-1/2}\, 2^{1-2\beta}\, \Gamma(\nu)}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha+\frac{1}{2}\nu\right)} \int_0^\infty h(x)$$

$$\times E\left[\Delta(m;\beta), \Delta(m;\beta-\nu+1) : \Delta\left(m;\beta+\frac{1}{2}\alpha-\frac{1}{2}\nu+\frac{1}{2}\right), \Delta\left(m;\beta-\frac{1}{2}\alpha-\frac{1}{2}\nu+1\right) : \frac{2^{2m}}{x}\right] dx,$$

provided the integrals are absolutely convergent, $R(\beta-\nu+1) > 0$, $R(\beta) > 0$.

Lemma 3. If

$$\Phi(p) \doteqdot h(x),$$

then

$$\text{(2.3)} \quad \int_0^{\pi/2} \exp\{i(\alpha+\beta-m)\theta\}(\sin\theta)^{\alpha-m-1}(\cos\theta)^{\beta-1}\Phi(e^{im\theta}\sin^m\theta)\, d\theta$$

$$= m^{-\beta} e^{i\frac{1}{2}\pi\alpha}\Gamma(\beta) \int_0^\infty h(x)\, E\left[\Delta(m;\alpha) : \Delta(m;\alpha+\beta) : e^{-i\frac{1}{2}\pi m}/x\right] dx,$$

provided the integrals are absolutely convergent, $R(\alpha) > 0$ and $R(\beta) > 0$.

Lemma 4. If

$$\Phi(p) \doteqdot h(x),$$

then

$$\text{(2.4)} \quad \int_0^{\pi/2} \exp\{i(\alpha+\beta-m-n)\theta\}(\sin\theta)^{\alpha-m-1}(\cos\theta)^{\beta-n-1} \times \Phi\{e^{i(m+n)\theta}\sin^m\theta\cos^n\theta\}\, d\theta$$

$$= e^{i\frac{1}{2}\pi\alpha} \sqrt{2\pi}\, \frac{m^{\alpha-1/2}\, n^{\beta-1/2}}{(m+n)^{\alpha+\beta-1/2}}$$

$$\times \int_0^\infty h(x)\, E\left[\Delta(m;\alpha),\, \Delta(n;\beta) \colon \Delta(m+n;\, \alpha+\beta) \colon \frac{(m+n)^{m+n}\, e^{-i\frac{1}{2}\pi m}}{m^m\, n^n\, x}\right] dx,$$

provided the integrals are absolutely convergent, $R(\alpha) > 0$ and $R(\beta) > 0$.

LEMMA 5. If

$$\Phi(p) \doteqdot h(x),$$

then

$$(2.5)\quad \int_0^1 \frac{t^{\alpha-m-1}\,(1-t)^{\beta-n-1}}{\{1+ct+d(1-t)\}^{\alpha+\beta-m-n}}\, \Phi\left[\frac{t^m\,(1-t)^n}{\{1+ct+d\,(1-t)\}^{m+n}}\right] dt,$$

$$= \frac{\sqrt{2\pi}\; m^{\alpha-1/2}\, n^{\beta-1/2}}{(1+c)^\alpha\,(1+d)^\beta\,(m+n)^{\alpha+\beta-1/2}}$$

$$\times \int_0^\infty h(x)\, E\left[\begin{matrix}\Delta(m;\alpha),\, \Delta(n;\beta) \\ \Delta(m+n;\, \alpha+\beta)\end{matrix} \colon \frac{(1+c)^m\,(1+d)^n\,(m+n)^{m+n}}{x\, m^m\, n^n}\right] dx,$$

provided the integrals are absolutely convergent, $R(\alpha) > 0$ and $R(\beta) > 0$.

PROOF OF LEMMA 1. By definition, we have

$$\Phi(p) = p \int_0^\infty e^{-px}\, h(x)\, dx$$

and therefore

$$\int_0^1 t^{\nu-m-1}\,(1-t)^{\nu-m-1}\, {}_2F_1\left(\alpha,\, \beta;\, \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2};\, t\right)$$

$$\times\, \Phi\,\{t^m\,(1-t)^m\}\, dt$$

$$= \int_0^1 t^{\nu-1}\,(1-t)^{\nu-1}\, {}_2F_1\left(\alpha,\, \beta;\, \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2};\, t\right)$$

$$\times \left[\int_0^\infty \exp\{-xt^m\,(1-t)^m\}\, h(x)\, dx\right] dt.$$

On interchanging the order of integration and evaluating the t-integral by (1.2), the result follows.

The interchange of the order of integration is permissible by virtue of the absolute convergence of the integrals involved in the above process.

The remaining LEMMAS can then be established in the same way on using (1.3), (1.4), (1.5) and (1.6) respectively.

3. APPLICATIONS. If we take

$$\Phi(p) = p\, G_{q,\, r}^{h,\, l}\left(z p \,\middle|\, \begin{matrix} a_1, \ldots, a_q \\ b_1, \ldots, b_r \end{matrix} \right)$$

we then have from author's formula [6, p. 40]

$$h(x) = x^{-1}\, G_{q+1,\, r}^{h,\, l}\left(\frac{z}{x} \,\middle|\, \begin{matrix} a_1, \ldots, a_q, 0 \\ b_1, \ldots, b_r \end{matrix} \right)$$

where $R(aj) < 1$ for $j = 1, 2, \ldots, l$; $R(p) > 0$,

$$|\arg z| < \left(h + l - \frac{1}{2} q - \frac{1}{2} r - \frac{1}{2} \right)\pi, \; h + l > \frac{1}{2}(q + r + 1).$$

Applying (2.1) we find that

$$\int_0^1 t^{\nu-1} (1-t)^{\nu-1}\, {}_2F_1\left(\alpha, \beta; \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2}; t \right)$$

$$\times\, G_{q,\, r}^{h,\, l}\left[z t^m (1-t)^m \,\middle|\, \begin{matrix} a_1, \ldots, a_q \\ b_1, \ldots, b_r \end{matrix} \right] dt$$

$$= \frac{\pi\, m^{-1/2}\, 2^{1-2\nu}\, \Gamma\left(\frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2} \right)}{z\, \Gamma\left(\frac{1}{2}\alpha + \frac{1}{2} \right) \Gamma\left(\frac{1}{2}\beta + \frac{1}{2} \right)}$$

$$\times \int_0^\infty G_{r,\, q+1}^{l,\, h}\left(\frac{x}{z} \,\middle|\, \begin{matrix} -b_1, \ldots, -b_r \\ -a_1, \ldots, -a_q, 0 \end{matrix} \right)$$

$$\times\, E\left[\Delta(m; \nu), \Delta\left(m; \nu - \frac{1}{2}\alpha - \frac{1}{2}\beta + \frac{1}{2} \right) : \Delta\left(m; \nu - \frac{1}{2}\alpha + \frac{1}{2} \right), \right.$$

$$\left. \Delta\left(m; \nu - \frac{1}{2}\beta + \frac{1}{2} \right) : \frac{2^{2m}}{x} \right] dx$$

Evaluating the integral on the right from MEIJER's formula [4, p. 82], it is seen that

$$(3.1)\qquad \int_0^1 t^{\nu-1}(1-t)^{\nu-1}\,{}_2F_1\left(\alpha,\ \beta;\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2};\ t\right)$$

$$\times\, G^{h,\ l}_{q,\ r}\left[zt^m(1-t)^m\,\middle|\,\begin{matrix}a_1,\ \dots,\ a_q\\ b_1,\ \dots,\ b_r\end{matrix}\right]\mathrm{dt}$$

$$=\frac{\pi\, m^{-1/2}\, 2^{1-2\nu}\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}$$

$$\times\, G^{h,\ l+2m}_{q+2m,\ r+2m}\left(\frac{z}{2^{2m}}\,\middle|\,\begin{matrix}\Delta(m;1-\nu),\Delta\left(m;\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\beta-\nu\right),\ a_1,\ \dots,\ a_q\\ b_1,\ \dots,\ b_r,\ \Delta\left(m;\frac{1}{2}+\frac{1}{2}\alpha-\nu\right),\Delta\left(m;\frac{1}{2}+\frac{1}{2}\beta-\nu\right)\end{matrix}\right).$$

The conditions of validity of (3.1) are as follows :

(*i*) $2(h+l)>q+r,\ |\arg z|<\left(h+l-\frac{1}{2}q-\frac{1}{2}r\right)\pi,$

$R(\nu+mb_j)>0,\ R\left(\frac{1}{2}+\nu+mb_j-\frac{1}{2}\alpha-\frac{1}{2}\beta\right)>0$ for $j=1,2,\dots,h$;

(*ii*) $q\geq r,\ 2(h+l)\geq q+r,\ |\arg z|\leq\left(h+l-\frac{1}{2}q-\frac{1}{2}r\right)\pi,$

$R(\nu+mb_j)>0,\ R\left(\frac{1}{2}+\nu+mb_j-\frac{1}{2}\alpha-\frac{1}{2}\beta\right)>0$ for $j=1,2,\dots,h,$

$$R\left[m\left(\sum_{i=1}^{q}a_i-\sum_{i=1}^{r}b_i-1/2\right)+(q-r)(\nu-m/2)\right]>-1,$$

$$R\left[m\left(\sum_{i=1}^{q}a_i-\sum_{i=1}^{r}b_i-1/2\right)+(q-r)\left(\nu+\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\beta-\frac{1}{2}m\right)\right]>-1$$

(*iii*) $q<r$ (or $q=r$ and $|z|<1$), $R(\nu+mb_j)>0,$

$R\left(1/2+\nu+mb_j-1/2\,\alpha-\frac{1}{2}\beta\right)>0$ for $j=1,2,\dots,h.$

The following results can be derived in the same way from Delete (2.2), (2.3), (2.4) and (2.5) respectively.

$$\text{(3.2)} \qquad \int_0^1 t^{\beta-1}(1-t)^{\beta-\nu}\, {}_2F_1(\alpha, 1-\alpha; \nu; t)$$

$$\times\, G^{h,\, l}_{q,\, r}\left[zt^m(1-t)^m \,\middle|\, \begin{matrix} a_1, \ldots, a_q \\ b_1, \ldots, b_r \end{matrix}\right] dt$$

$$= \frac{\pi\, m^{-1/2}\, 2^{1-2\beta}\, \Gamma(\nu)}{\Gamma\left(\frac{1}{2}\alpha + \frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\alpha\right)}$$

$$\times\, G^{h,\, l+2m}_{q+2m,\, r+2m}\left(\frac{z}{2^{2m}} \,\middle|\, \begin{matrix} \Delta(m; 1-\beta), \Delta(m; \beta-\nu), a_1, \ldots, a_q \\ b_1, \ldots, b_r, \Delta\left(m; \frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha-\beta\right), \Delta\left(m; \frac{1}{2}\alpha+\frac{1}{2}\nu-\beta\right) \end{matrix}\right).$$

(3.2) is valid under the following sets of conditions:

(*i*) $\qquad 2(h+l) > q+r, \; |\arg z| < \left(h+l-\frac{1}{2}q-\frac{1}{2}r\right)\pi,$

$R(\beta + mb_j) > 0, \; R(1+\beta-\nu+mb_j) > 0$ for $j = 1, 2, \ldots, h$;

(*ii*) $\qquad q \geq r, \; 2(h+l) \geq q+r, \; |\arg z| \leq \left(h+l-\frac{1}{2}q-\frac{1}{2}r\right)\pi$

$R(\beta + mb_j) > 0, \; R(1+\beta-\nu+mb_j) > 0$ for $j = 1, 2, \ldots, h,$

$$R\left[m\left(\sum_{i=1}^{q} a_i - \sum_{i=1}^{r} b_i - 1/2\right) + (q-r)\left(\beta - \frac{1}{2}m\right)\right] > -1,$$

$$R\left[m\left(\sum_{i=1}^{q} a_i - \sum_{i=1}^{r} b_i - 1/2\right) + (q-r)\;(\beta - \nu - m/2 + 1)\right] > -1;$$

(*iii*) $\qquad q < r$ (or $q = r$ and $|z| < 1$), $R(\beta + mb_j) > 0,$

$R(1+\beta-\nu+mb_j) > 0$ for $j = 1, 2, \ldots, h$;

(3.3)
$$\int_0^{\pi/2} \exp\{i(\alpha+\beta)\theta\}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}$$
$$\times G^{h,\,l}_{q,\,r}\left[ze^{im\theta}\sin^m\theta \,\middle|\, \begin{matrix} a_1, \ldots, a_q \\ b_1, \ldots, b_r \end{matrix}\right] d\theta$$
$$= e^{i\frac{1}{2}\pi\alpha}\, m^{-\beta}\, \Gamma(\beta)$$
$$\times G^{h,\,l+m}_{q+m,\,r+m}\left(z e^{i\frac{1}{2}\pi m} \,\middle|\, \begin{matrix} \Delta(m;\,1-\alpha),\, a_1, \ldots, a_q \\ b_1, \ldots, b_r,\, \Delta(m;\,1-\alpha-\beta) \end{matrix}\right).$$

This formula is valid under the following sets of conditions:

(*i*) $2(h+l) > q+r$, $|\arg z| < \left(h+l-\frac{1}{2}q-\frac{1}{2}r\right)\pi$,

$R(\beta) > 0$, $R(\alpha + mb_j) > 0$ for $j = 1, 2, \ldots, h$;

(*ii*) $q \geq r$, $2(h+l) \geq q+r$, $|\arg z| \leq \left(h+l-\frac{1}{2}q-\frac{1}{2}r\right)\pi$

$R(\alpha + mb_j) > 0$ for $j = 1, 2, \ldots, h$, $R(\beta) > 0$,

$R\left[m\left(\sum_{j=1}^{q} a_j - \sum_{j=1}^{r} b_j - {}^1/_2\right) + (q-r)\,(\alpha - {}^m/_2)\right] > -1$,

(*iii*) $q < r$ (or $q = r$ and $|z| < 1$), $R(\beta) > 0$ and

$R(\alpha + mb_j) > 0$ for $j = 1, 2, \ldots, h$.

(3.4)
$$\int_0^{\pi/2} \exp\{i(\alpha+\beta)\theta\}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}$$
$$\times G^{h,\,l}_{q,\,r}\left[z\, e^{i(m+n)\theta}\sin^m\theta\,\cos^n\theta \,\middle|\, \begin{matrix} a_1, \ldots, a_q \\ b_1, \ldots, b_r \end{matrix}\right] d\theta$$
$$= e^{i\frac{1}{2}\pi\alpha}\sqrt{2\pi}\,\frac{m^{\alpha-1/2}\, n^{\beta-1/2}}{(m+n)^{\alpha+\beta-1/2}}$$
$$\times G^{h,\,l+m+n}_{q+m+n,\,r+m+n}\left[\frac{zm^m n^n e^{i\frac{1}{2}\pi m}}{(m+n)^{m+n}} \,\middle|\, \begin{matrix} \Delta(m;\,1-\alpha),\, \Delta(n;\,1-\beta),\, a_1, \ldots, a_q \\ b_1, \ldots, b_r,\, \Delta(m+n;\,1-\alpha-\beta) \end{matrix}\right]$$

and

$$(3.5)\quad \int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{\{1+ct+d(1-t)\}^{\alpha+\beta}} G^{h,\, l}_{q,\, r}\left[\frac{z\, t^m (1-t)^n}{\{1+ct+d(1-t)\}^{m+n}}\,\middle|\, \begin{matrix} a_1, \dots, a_q \\ b_1, \dots, b_r \end{matrix}\right] dt$$

$$= \frac{\sqrt{2\pi}\; m^{\alpha-1/2}\; n^{\beta-1/2}}{(1+c)^\alpha\,(1+d)^\beta\,(m+n)^{\alpha+\beta-1/2}}$$

$$\times G^{h,\; l+m+n}_{q+m+n,\, r+m+n}\left[\frac{z\, m^m\, n^n}{(1+c)^m (1+d)^n (m+n)^{m+n}}\,\middle|\, \begin{matrix} \Delta(m; 1-\alpha), \Delta(n; 1-\beta), a_1, \dots, a_q \\ b_1, \dots, b_r, \Delta(m+n; 1-\alpha-\beta) \end{matrix}\right]$$

where c and d satisfy the conditions of formula (1.6).

The conditions of validity of (3.4) and (3.5) are given below.

(i) $\quad 2(h+l) > q+r, \ |\arg z| < \left(h+l-\frac{1}{2}q-\frac{1}{2}r\right)\pi,$

$$R(\alpha+mb_j) > 0,\ R(\beta+nb_j) > 0 \text{ for } j = 1, 2, \dots, h;$$

(ii) $\quad q \geq r,\ 2(h+l) \geq q+r,\ |\arg z| \leq \left(h+l-\frac{1}{2}q-\frac{1}{2}r\right)\pi,$

$$R(\alpha+mb_j) > 0,\ R(\beta+nb_j) > 0 \text{ for } j = 1, 2, \dots, h,$$

$$R\left[m\left(\sum_{i=1}^{q} a_i - \sum_{i=1}^{r} b_i - 1/2\right) + (q-r)(\alpha - m/2)\right] > -1,$$

$$R\left[n\left(\sum_{i=1}^{q} a_i - \sum_{i=1}^{r} b_i - 1/2\right) + (q-r)(\beta - n/2)\right] > -1;$$

(iii) $\quad q < r$ (or $q = r$ and $|z| < 1$),

$$R(\alpha+mb_j) > 0.\ R(\beta+nb_j) > 0 \text{ for } j = 1, 2, \dots, h.$$

MacRobert's formulae [2] and [3] can be derived from the results (3.1) to (3.5), on taking $h = 1$, $l = q = \nu$, $r = \delta + 1$, $b_1 = 0$ writing $1 - b_{i-1}$ and $1 - a_j$ for the parameters b_i and a_j ($i = 2, 3, \dots, \delta + 1$ and $j = 1, 2, \dots, \nu$) replacing z by $1/z$ and using the identity [1, p. 215]

$$(3.6)\quad G^{1,\, \nu}_{\nu,\, \delta+1}\left(x\,\middle|\, \begin{matrix} 1-a_1, \dots, 1-a_\nu \\ 0, 1-b_1, \dots, 1-b_\delta \end{matrix}\right) = E(\nu; a_i: \delta; b_j: 1/x).$$

The author is grateful to Dr. R. S. Varma for his valuable suggestions in the preparation of this paper.

REFERENCES

1. Bateman Manuscript Project. — *Higher transcendental functions*. Vol. 1- McGraw-Hill; New York (1953).

2. MacRobert, T. M., *Integrals involving hypergeometric functions and E-functions*, Proc. Glasgow Math. Assoc., Vol. **3** (1958), 196-198.

3. MacRobert, T. M., *Beta function formulae and integrals involving E-functions*, Math. Annalen, Vol. **142** (1961), p. 450-452.

4. Meijer, C. S., *Neue Integraldarstellungen für Whittakersche Funcktionen* Proc. Nederl. Akad. Wetenchappen (Amsterdam). Vol. **44** (1941), p. 81-92.

5. Meijer, C. S., *On the G-function, VI, VII*, Proc. Nederl. Akad. Wetenschappen (Amsterdam). Vol. **49** (1946), pp. 936-943, 1063-1072.

6. Saxena, R. K., *Some theorems in operational calculus and infinite integrals involving Bessel function and G-functions*, Proc. Nat. Inst. Sci., India, Vol. **27** (1961), p. 38-61.

7. Saxena, R. K., *Some formulae for the G-function*, Proc. Cambridge Philos. Soc. Vol. **59** (1963), p. 347-350.

Faculty of Engineering
Department of Mathematics
University of Jodhpur
Jodhpur, India