PRODUCTS OF APPELL POLYNOMIALS

by

L. CARLITZ

1. Let

$$(1.1) \qquad f(t) = \sum_0^\infty f_n \frac{t^n}{n!} \qquad (f_0 = 1)$$

denote a formal power series and put

$$(1.2) \qquad f(t) e^{xt} = \sum_0^\infty f_n(x) \frac{t^n}{n!} \,,$$

so that

$$(1.3) \qquad f_n(x) = \sum_{r=0}^n \binom{n}{r} f_{n-r} x^n, \; f_n(0) = f_n \,.$$

Thus with every power series $f(t)$ is associated the set of APPELL plynomials $\{f_n(x)\}$ defined by means of (1.2) or (1.3). The polynomials $f_n(x)$ have the property.

$$(1.4) \qquad f'_n(x) = n f_{n-1}(x) \; ;$$

conversely if a set of polynomials have the property (1.4) then there exists a power series $f(t)$ such that (1.2) holds.

It is convenient to put

$$(f(t))^{-1} = \sum_0^\infty f'_n \frac{t^n}{n!} \qquad (f'_0 = 1),$$

so that

$$(1.5) \qquad x^n = \sum_{r=0}^n \binom{n}{r} f'_{n-r} f_r(x).$$

In addition to $f(t)$ we also take

$$g(t) = \sum_0^\infty g_n \frac{t^n}{n!}, \; h(t) = \sum_0^\infty h_n \frac{t^n}{n!} \; (g_0 = h_0 = 1)$$

and put

$$g(t) e^{xt} = \sum_0^\infty g_n(x) \frac{t^n}{n!}, \; h(t) e^{xt} = \sum_0^\infty h_n(x) \frac{t^n}{n!}.$$

Now consider

$$f_m(ax) g_n(bx) = \sum_{r=0}^m \sum_{s=0}^n \binom{m}{r} \binom{n}{s} f_{m-r} g_{n-s} a^r b^s x^{r+s}.$$

As in (1.5)

$$x^n = \sum_{r=0}^n \binom{n}{r} h'_{n-r} h_r(x),$$

where

(1.6) $$\qquad\qquad (h(t))^{-1} = \sum_0^\infty h'_n \frac{t^n}{n!}.$$

Thus

$$f_m(ax) g_n(bx) = \sum_{r=0}^m \sum_{s=0}^n \binom{m}{r} \binom{n}{s} f_{m-r} g_{n-r} a^r b^s \cdot \sum_{j=0}^{r+s} \binom{r+s}{j} h'_{r+s-j} h_j(x).$$

We put

(1.7) $$\qquad\qquad f_m(ax) g_n(bx) = \sum_{j=0}^{m+n} A_j^{(m,n)} h_j(x),$$

where

(1.8) $$A_j^{(m,n)} = A_j^{(m,n)}(a,b) = \sum_{r=0}^m \sum_{s=0}^n \binom{m}{r}\binom{n}{s}\binom{r+s}{j} f_{m-r} g_{n-s} h'_{r+s-j} a^r b^s.$$

Then

$$\sum_{m,n=0}^\infty A_j^{(m,n)} \frac{u^m v^n}{m!n!} = \sum_{m,n=0}^\infty \frac{u^m v^n}{m!n!} \sum_{r=0}^m \sum_{s=0}^n \binom{m}{r}\binom{n}{s}\binom{r+s}{j} f_{m-r} g_{n-s} h'_{r+s-j} a^r b^s$$

$$= \sum_{r,s=0}^\infty \frac{(au)^r (bv)^s}{r!s!} \binom{r+s}{j} h'_{r+s-j} \cdot \sum_{m,n=0}^\infty f_m g_n \frac{u^m v^n}{m!n!}$$

$$= f(u)g(v) \sum_{r,s=0}^\infty \frac{(au)^r (bv)^s}{r!s!} \binom{r+s}{j} h'_{r+s-j}$$

$$= f(u) g(v) \sum_{k=j}^{\infty} \binom{k}{j} h'_{k-j} \sum_{r+s=k} \frac{(au)^r (bv)^s}{r! \, s!}$$

$$= f(u) g(v) \sum_{k=j}^{\infty} \binom{k}{j} h'_{k-j} \frac{(au+bv)^k}{k!}$$

$$= f(u) g(v) \sum_{k=0}^{\infty} h'_k \frac{(au+bv)^{k+j}}{j! \, k!} = \frac{f(u) g(v)}{h(au+bv)} \frac{(au+bv)^j}{j!}$$

by (1.6). We have therefore obtained

$$(1.9) \qquad \sum_{m,n=0}^{\infty} A_j^{(m,n)} \frac{u^m v^n}{m! \, n!} = \frac{f(u) g(v)}{h(au+bv)} \frac{(au+bv)^j}{j!} \,,$$

a generating function for $A_j^{(m,n)}$ with j fixed.

We remark that there is no difficulty in proving the corresponding formula for the product of an arbitrary number of APPELL polynomials. If

$$f^{(i)}(t) = \sum_{n=0}^{\infty} f_n^{(i)} \frac{t^n}{n!}, \quad f_0^{(i)} = 1 \qquad (i = 1, \ldots, k),$$

$$f^{(i)}(t) e^{xt} = \sum_{n=0}^{\infty} f_n^{(i)}(x) \frac{t^n}{n!}$$

and we put

$$(1.10) \qquad f_{m_1}^{(1)}(a_1 x) \ldots f_{m_k}^{(k)}(a_k x) = \sum_{j=0}^{m_1+\ldots+m_k} A_j^{(m_1,\ldots,m_k)} h_j(x),$$

then we find that

$$(1.11) \qquad \sum_{m_1,\ldots,m_k=0}^{\infty} A_j^{(m_1,\ldots,m_k)} \frac{u_1^{m_1} \ldots u_k^{m_k}}{m_1! \ldots m_k!}$$

$$= \frac{f^{(1)}(u_1) \ldots f^{(k)}(u_k)}{h(a_1 u_1 + \ldots + a_k u_k)} \frac{(a_1 u_1 + \ldots + a_k u_k)^j}{j!} \,.$$

2. In place of (1.7) we may write

$$(2.1) \qquad f_m(ax) g_n(bx) = \sum_{j=0}^{m+n} A_{m+n-j}^{(m,n)} h_{m+n-j}(x).$$

This suggests evaluating

$$\sum_{m,n=0}^{\infty} A_{m+n-j}^{(m,n)} \frac{u^m v^n}{m! \, n!} \,.$$

By (1.8) this sum is equal to

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!\,n!} \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r}\binom{n}{s}\binom{r+s}{m+n-j} f_{m-r}\,g_{n-s}\cdot h'_{r+s-m-n+j}\,a^r b^s$$

$$= \sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!\,n!} \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r}\binom{n}{s}\binom{m+n-r-s}{j-r-s} f_r\,g_s\,h'_{j-r-s}\cdot a^{m-r} b^{n-s}$$

$$= \sum_{r+s\leq j} f_r\,g_s\,h'_{j-r-s}\,\frac{u\,v^s}{r!\,s!}\cdot \sum_{m,n=0}^{\infty}\binom{m+n}{j-r-s}\frac{(au)^m\,(bv)^n}{m!\,n!}$$

$$= \sum_{r+s\leq j} f_r\,g_s\,h'_{j-r-s}\,\frac{u\,v^s}{r!\,s!}\cdot \sum_{m=0}^{\infty}\binom{m}{j-r-s}\frac{(au+bv)^m}{m!}$$

$$= \sum_{r+s\leq j} f_r\,g_s\,\frac{h'_{j-r-s}}{(j-r-s)!}\,\frac{u^r\,v^s}{r!\,s!}\sum_{m=0}^{\infty}\frac{(au+bv)^{m+j-r-s}}{m!}.$$

We have therefore proved

(2.2) $$\sum_{m,n=0}^{\infty} A^{(m,n)}_{m-n-j}\,\frac{u^m v^n}{m!\,n!} = e^{au+bv}\sum_{r+s\leq j} f_r\,g_s\,h'_{j-r-s}\,\frac{u^r\,v^s}{r!\,s!}\,\frac{(au+bv)^{j-r-s}}{(j-r-s)!}.$$

This suggests that we multiply by t^j and sum over j. Then

$$\sum_{j=0}^{\infty} t^j \sum_{r+s\leq j} f_r\,g_s\,h'_{j-r-s}\,\frac{u^r\,v^s}{r!\,s!}\,\frac{z^{j-r-s}}{(j-r-s)!}$$

$$= \sum_{r,s=0}^{\infty} f_r\,g_s\,\frac{(tu)^r\,(tv)^s}{r!\,s!}\sum_{j=r+s}^{\infty} h'_{j-r-s}\,\frac{(tz)^{j-r-s}}{(j-r-s)!}$$

$$= \frac{1}{h(tz)}\sum_{r,s=0}^{\infty} f_r\,g_s\,\frac{(tu)^r\,(tv)^s}{r!\,s!} = \frac{f(tu)\,g(tv)}{h(tz)}.$$

We have therefore

(2.3) $$\sum_{m,n,j=0}^{\infty} A^{(m,n)}_{m+n-j}\,\frac{u^m v^n}{m!\,n!}\,t^j = e^{au+bv}\,\frac{f(tu)\,g(tv)}{h(t(au+bv))}.$$

The generalization corresponding to (1.11) will be omitted.
Another variant may be mentioned. Rewrite (2.1) in the form

(2.4) $$f_m(ax)\,g_n(bx) = \sum_{j=0}^{m+n} B_j^{(m,n)}\,\frac{h_{m+n-j}(x)}{(m+n-j)!},$$

so that

$$B_j^{(m,n)} = (m+n-j)!\,A^{(m,n)}_{m+n-j}.$$

Making use of (1.8) we get

$$B_j^{(m,\,n)} = \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} \binom{m+n-r-s}{j-r-s} (m+n-j)! \cdot f_r\, g_s\, h'_{j-r-s}\, a^{m-r}\, b^{n-s}$$

$$= \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} \frac{(m+n-r-s)!}{(j-r-s)!}\, f_r\, g_s\, h'_{j-r-s} \cdot a^{m-r}\, b^{n-s}.$$

This evidently defines $B_j^{(m,\,n)}$ for all m, n, j; the condition $m+n \geq j$ is no longer needed. We now find that

$$(2.5) \qquad \sum_{m,\,n=0}^{\infty} B_j^{(m,\,n)} \frac{u^m v^n}{m!\, n!} = \frac{1}{1-au-bv} \sum_{r+s\leq j} f_r\, g_s\, \frac{h'_{j-r-s}}{(j-r-s)!} \frac{u^r v^s}{r!\, s!}$$

and

$$(2.6) \qquad \sum_{m,\,n,\,j=0}^{\infty} B_j^{(m,\,n)} \frac{u^m v^n}{m!\, n!}\, t^j = \frac{1}{1-au-bv} \frac{f(tu)\, g(tv)}{h(t)}\,.$$

This formula admits of an obvious generalization analogous to (1.11).

3. There are many well known instances of APPELL polynomials. The BERNOULLI and EULER polynomials defined by means of

$$(3.1) \qquad \frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

and

$$(3.2) \qquad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}\,,$$

respectively, furnish examples; more generally this is also true of the BERNOULLI and EULER polynomials of higher order [4, Ch. 5] which may be defined by

$$(3.3) \qquad \left(\frac{t}{e^t-1}\right)^z e^{xt} = \sum_{n=0}^{\infty} B_n^{(z)}(x) \frac{t^n}{n!}\,,$$

$$(3.4) \qquad \left(\frac{2}{e^t-1}\right)^z e^{xt} = \sum_{n=0}^{\infty} E_n^{(z)}(x) \frac{t^n}{n!}\,,$$

where z is an arbitrary complex number. We may also cite NÖR-LUND's polynomials of order k defined by

$$(3.5) \qquad \frac{\omega_1 \ \cdots \ \omega_k \ t^k \ c^{xt}}{(e^{\omega_1 t} - 1) \ \cdots \ (e^{\omega_k t} - 1)} = \sum_{n=0}^{\infty} B_n^{(k)} (x \mid \omega_1, \ \ldots, \ \omega_k) \frac{t^n}{n!},$$

$$(3.6) \qquad \frac{2^k \ e^{xt}}{(e^{\omega_1 t} + 1) \ \cdots \ (e^{\omega_k t} + 1)} = \sum_{n=0}^{\infty} E_n^{(k)} (x \mid \omega_1, \ \ldots, \ \omega_k) \frac{t^n}{n!}.$$

A less familiar example is that of the « EULERIAN » polynomial [1]

$$\phi_n (x) = \phi_n (x \mid \lambda)$$

defined by

$$(3.7) \qquad \frac{1 - \lambda}{e^x - \lambda} e^{xt} = \sum_{n=0}^{\infty} \phi_n (x \mid \lambda) \frac{t^n}{n!} \quad (\lambda \neq 1).$$

This polynomial can be generalized along the lines of (3.4) and (3.6).

In the next place the HERMITE polynomial $H_n (x)$ may be mentioned. We have the familiar generating function

$$(3.8) \qquad e^{2xt - t^2} = \sum_{n=0}^{\infty} H_n (x) \frac{t^n}{n!};$$

thus the polynomials $\{ H_n (x/2) \}$ form an APPELL set. In this case we evidently have

$$f(t) = e^{-t^2},$$

so that

$$f_n = \begin{cases} (-1)^r \dfrac{(2r)!}{r!} & (n = 2r) \\ & (n \ odd). \end{cases}$$

If we take

$$(3.9) \qquad f(t) = g(t) = h(t) = e^{-t^2}$$

we find that (2.2) and (2.5) can be simplified. We may assume j even. Then

$$\sum_{r+s \leq j} f_{2r} \ g_{2s} \ h'_{2j-2r-2s} \frac{u^{2r} \ v^{2s}}{(2r)! \ (2s)!} \frac{(au + bv)^{2j-2r-2s}}{(2j - 2r - 2s)!}$$

$$= \sum_{r+s \leq j} (-1)^{r-s} \frac{u^{2r} \ v^{2s}}{r! \ s!} \frac{(au + bv)^{2j-2r-2j}}{(j - r - s)!} = \frac{1}{j!} (1 - a^2 \ u^2 - b^2 \ v^2)^j.$$

Thus (2.2) becomes

$$(3.10) \qquad \sum_{m,n=0}^{\infty} A_{m+n-2j}^{(m,n)} \frac{u^m v^n}{m!\,n!} = \frac{1}{j!} (1 - a^2 u^2 - b^2 v^2)^j e^{au+bv}.$$

Similarly we find that, when (3.9) is satisfied (2.5) reduces to

$$(3.11) \qquad \sum_{m,n=0}^{\infty} B_{2j}^{(m,n)} \frac{u^m v^n}{m!\,n!} = \frac{1}{j!} \frac{(1 - a^2 u^2 - b^2 v^2)^j}{1 - au - bv}.$$

We remark that (3.11) was derived in [3] for the product of an arbitrary number of HERMITE polynomials.

4. For the LAGUERRE polynomial $L_n^{(\alpha)}(x)$ we have the two generating functions

$$(4.1) \qquad \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n = (1-t)^{-\alpha-1}\, e^{-xt/(1-t)},$$

$$(4.2) \qquad \sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)\, t^n = (1+t)^{\alpha}\, e^{-xt}.$$

Thus the polynomials $\{L_n^{(\alpha-n)}(-x)\}$ constitute an APPELL set and therefore the results of §§ 1,2 apply. This is however not true of the $\{L_n^{(\alpha)}(x)\}$. We therefore consider the following situation.

Let $f(t)$ be of the form (1.1) and let

$$F(t) = \sum_{n=1}^{\infty} F_r t^r$$

be a formal power series without constant term. Put

$$(4.3) \qquad f(t)\, e^{xF(t)} = \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{n!}.$$

We also put

$$g(t)\, e^{xG(t)} = \sum_{n=0}^{\infty} g_n(x) \frac{t^n}{n!}, \quad h(t)\, e^{xH(t)} = \sum_{n=0}^{\infty} h_n(x) \frac{t^n}{n!},$$

where $G(t)$, $H(t)$ are series similar to $F(t)$.

Let

$$f_n(x) = \sum_{r=0}^{n} \binom{n}{r} f_{n,r}\, x^r, \quad g_n(x) = \sum_{r=0}^{n} \binom{n}{r} g_{n,r}\, x^r, \quad h_n(x) = \sum_{r=0}^{\infty} \binom{n}{r} h_{n,r}\, x^r.$$

Then we have

(4.4) $$f_m(ax)\,g_n(bx) = \sum_{j=0}^{m+n} C_j^{(m,\,n)}\,h_j(x)\,,$$

where

(4.5) $$C_j^{(m,\,n)} = \sum_{r=0}^{m}\sum_{s=0}^{n} \binom{m}{r}\binom{n}{s}\binom{r+s}{j}\,a^r\,b^s\,.\,f_{m,\,r}\,g_{n,\,s}\,h'_{r+s,\,j}$$

and

(4.6) $$x^n = \sum_{j=0}^{n} \binom{n}{j}\,h'_{n,\,j}\,h_j(x)\,.$$

It follows that

(4.7) $$\sum_{m,\,n=0}^{\infty} C_j^{(m,\,n)}\,\frac{u^m\,v^n}{m!\,n!} = \sum_{r,\,s=0}^{\infty} \frac{(au)^r\,(bv)^s}{r!\,s!}\binom{r+s}{j}\,h'_{r-s,\,j}$$

$$\cdot \sum_{m,\,n=0}^{\infty} f_{m+r,\,r}\,g_{n+s,\,s}\,\frac{u^m\,v^n}{m!\,n!}\,.$$

On the other hand, we have

$$\sum_{m=0}^{\infty} f_m(x)\,\frac{t^m}{m!} = \sum_{m=0}^{\infty} \frac{t^m}{m!}\sum_{r=0}^{m}\binom{m}{r}\,f_{m,\,r}\,x^r = \sum_{r=0}^{\infty}\frac{(xt)^r}{r!}\sum_{m=0}^{\infty} f_{m+r,\,r}\,\frac{t^m}{m!}$$

$$= \sum_{r=0}^{\infty}\frac{(xt)^r}{r!}\,f_r(t)\,,$$

say. But since by (4.3)

$$\sum_{m=0}^{\infty} f_m(x)\,\frac{t^m}{m!} = f(t)\,e^{xF(t)} = \sum_{r=0}^{\infty} f(t)\,F^r(t)\,\frac{x^r}{r!}\,,$$

it is clear that

(4.8) $$t^r\,f_r(t) = f(t)\,F^r(t)\,.$$

Thus (4.7) becomes

(4.9) $$\sum_{,\,n=0}^{\infty} C_j^{(m,\,n)}\,\frac{u^m\,v^n}{m!\,n!} = \sum_{r,\,s=0}^{\infty} \frac{a^r\,b^s}{r!\,s!}\binom{r+s}{j}\,h'_{r+s,\,j}\cdot f(u)F^r(u)\,.\,g(v)G^s(v)$$

$$= f(u)\,g(v)\sum_{k=j}^{\infty}\binom{k}{j}\,h'_{k,\,j}\sum_{r+s=k} \frac{a^r\,b^s\,F^r(u)\,G^s(v)}{r!\,s!}$$

$$= f(u)\,g(v) \sum_{k=j}^{\infty} \binom{k}{j} h'_{k,j} \frac{(a\,F(u) + b\,G(v))^k}{k!}$$

$$= f(u)\,g(v) \frac{(a\,F(u) + b\,G(v))^j}{j!} \cdot \sum_{k=0}^{\infty} h'_{k+j,j} \frac{(a\,F(u) + b\,G(v))^k}{k!} .$$

We now make an additional assumption concerning

$$H(t) = \sum_{n=1}^{\infty} H_n\,t^n ,$$

namely $H_1 \ne 0$. This is equivalent to assuming the existence of a series $I(t)$ such that

(4.10) $$H(I(t)) = I(H(t)) = t .$$

Using (4.6) we get

$$e^{xH(t)} = \sum_{j=0}^{\infty} h_j(x) \frac{H^j(t)}{j!} \sum_{n=0}^{\infty} h'_{n+j,j} \frac{H^n(t)}{n!} .$$

Since

$$e^{xH(t)} = \frac{1}{h(t)} \sum_{j=0}^{\infty} h_j(x) \frac{t^j}{j!} .$$

it follows that

$$H^j(t) \sum_{n=0}^{\infty} h'_{n+j,j} \frac{H^n(t)}{n!} = \frac{t^j}{h(t)} .$$

Now using (4.10) it follows that

$$t^j \sum_{n=0}^{\infty} h'_{n+j,j} \frac{t^n}{n!} = \frac{I^j(t)}{h(I(t))} .$$

Thus (4.9) becomes

(4.11) $$\sum_{m,n=0}^{\infty} C_j^{(m,n)} \frac{u^m\,v^n}{m!\,n!} = \frac{f(u)\,g(v)}{h\{I(aF(u)+bG(v))\}} \frac{I^j(a\,F(u)+bG(v))}{j!} .$$

The generalization corresponding to a product of k polynomials is immediate.

We now examine the special case suggested by (4.1), that is

$$H(t) = \frac{t}{1-t} \quad , \quad I(t) = \frac{t}{1+t} .$$

If in addition we take

$$h(t) = (1 - t)^{-\alpha-1},$$

we find that the right member of (4.11) reduces to

(4.12) $$\frac{f(u)\,g(v)}{j!} \frac{(a\,F(u) + bG(v))^j}{(1 + a\,F(u) + bG(v))^{\alpha+j+1}},$$

which is in agreement with [3, (6)].

5.   Finally we discuss briefly a q-analog of (1.9). Put

(5.1) $$e(t) = \prod_0^\infty (1 - q^n\, t)^{-1} = \sum_0^\infty \frac{t^n}{(q)_n},$$

where

$$(q)_n = (1 - q)\,(1 - q^2)\,\ldots\,(1 - q^n).$$

Let

(5.2) $$f(t)\,e(xt) = \sum_0^\infty f_n(x)\,\frac{t^n}{(q)_n},$$

where

$$f(t) = \sum_0^\infty f_n\,\frac{t^n}{(q)_n} \qquad (f_0 = 1);$$

then

(5.3) $$f_n(x) = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} f_{n-r}\,x^r,$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(q)_n}{(q)_r\,(q)_{n-r}}.$$

We also put

(5.4) $$\frac{1}{f(t)} = \sum_0^\infty f'_n\,\frac{t^n}{(q)_n} \qquad (f'_0 = 1),$$

so that

(5.5) $$x^n = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} f'_{n-r}\,f_n(x).$$

Now let $g(t)$, $h(t)$ be two other series of the same nature as $f(t)$ and define $g_n(x)$, $h_n(x)$, g'_n, h'_n by formulas like (5.4) and (5.5).

Then if we put

$$(5.6) \qquad f_m(ax)\, g_n(bx) = \sum_{j=0}^{m+n} A_j^{(m,\,n)}\, h_j(x)\,,$$

we find that

$$(5.7) \qquad A_j^{(m,\,n)} = \sum_{r=0}^{m} \sum_{s=0}^{n} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} \begin{bmatrix} r+s \\ j \end{bmatrix} \cdot f_{m-r}\, g_{n-s}\, h'_{r+s-j}\, a^r\, b^s\,.$$

Thus

$$\sum_{m,\,n=0}^{\infty} A_j^{(m,\,n)} \frac{u^m\, v^n}{(q)_m\, (q)_n} = \sum_{r,\,s=0}^{\infty} \frac{(au)^r\, (bv)^s}{(q)_r\, (q)_s} \begin{bmatrix} r+s \\ j \end{bmatrix} h'_{r+s-j}$$

$$\cdot \sum_{m,\,n=0}^{\infty} f_m\, g_n\, \frac{u^m\, v^n}{(q)_m\, (q)_n}\,.$$

We have therefore

$$(5.8) \qquad \sum_{m,\,n=0}^{\infty} A_j^{(m,\,n)} \frac{u^m\, v^n}{(q)_m\, (q)_n} = f(u)\, g(v) \sum_{r,\,s=0}^{\infty} \frac{(au)^r\, (bv)^s}{(q)_r\, (q)_s} \begin{bmatrix} r+s \\ j \end{bmatrix} h'_{r+s-j}.$$

If we put

$$(5.9) \qquad H_k(a,\, b) = \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix} a^r\, b^{k-r}\,,$$

the sum on the right of (5.8) becomes

$$(5.10) \qquad \frac{1}{(q)_j} \sum_{k=0}^{\infty} \frac{h'_k}{(q)_k}\, H_{k+j}(au,\, bv)\,.$$

This does not seem to admit of further simplification. If we multiply both sides of (5.8) by t^j and sum over j, we find similarly that

$$(5.11) \qquad \sum_{m,n,j=0}^{\infty} A_j^{(m,\,n)} \frac{u^m\, v^n}{(q)_m\, (q)_n}\, t = f(u)\, g(v) \sum_{n=0}^{\infty} \frac{1}{(q)_n}\, h'_n(t)\, H_n(au,\, bv)\,,$$

where

$$(5.12) \qquad \frac{e(xt)}{h(t)} = \sum_{0}^{\infty} h'_n(x) \frac{t^n}{(q)_n}\,.$$

If we take $h(t) = (e\,(t))^{-1}$, then since

$$\sum_{k=0}^{\infty} \frac{1}{(q)_k} H_k\,(a,\,b) = e\,(a)\,e\,(b)\,,$$

it follows that

(5.13) $h'_n\,(x) = H_n\,(x,\,1)\,.$

Incidentally, since

$$\frac{e\,(xt)}{e\,(t)} = \sum_{n=0}^{\infty} \frac{(x-1)\,(x-q)\ldots(x-q^{n-1})}{(q)_n}\,t^n\,,$$

it is clear that in the present case

$$h_n\,(x) = (x-1)(x-q)\ldots(x-q^{n-1}).$$

In view of (5.13), the right member of (5.11) is equal to

$$f\,(u)\,g\,(v)\sum_{n=0}^{\infty} \frac{(au)^n}{(q)_n}\,H_n\,(t,\,1)\,H_n\left(\frac{bv}{an},\,1\right).$$

The writer has proved the identity [2, (3.9)]

(5.14) $\displaystyle\sum_{0}^{\infty} \frac{t^n}{(q)_n}\,H_n\,(a,\,1)\,H_n\,(b,\,1) = \frac{e\,(t)\,e\,(at)\,e\,(bt)\,e\,(abt)}{e\,(abt^2)}$

Thus (5.11) becomes

(5.15) $\displaystyle\sum_{m,n,j=0}^{\infty} A_j^{(m,n)}\frac{u^m\,v^n}{(q)_m(q)_n}\,t^j = f\,(u)\,g\,(v)\frac{e\,(au)\,e\,(bv)\,e\,(aut)\,e\,(bvt)}{e\,(abuvt)}\,.$

While (5.8) and (5.11) can be generalized for the product of an arbitrary number of polynomials, it does not seem possible to do the same for (5.15) unless a suitable generalization of (5.14) can be obtained.

Interesting variants of the above results can be obtained if we make a slight change in the definition of $g_n(x)$ but leave all other quantities unchanged. Put

(5.16) $\displaystyle\frac{g\,(t)}{e\,(xt)} = \sum_{0}^{\infty} \bar{g}_n\,(x)\,\frac{t^n}{(q)_n}\,.$

Since

$$\frac{1}{e\,(t)} = \prod_{0}^{\infty} (1 - q^n\,t) = \sum_{0}^{\infty} (-1)^n\,q^{\frac{1}{2}n\,(n-1)}\,\frac{t^n}{(q)_n}\,,$$

it follows that

$$\bar{g}_n(x) = \sum_{s=0}^{n} (-1)^s \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}s(s-1)} g_{n-s} x^s .$$

If we now put

$$f_m(ax) \bar{g}_n(bx) = \sum_{j=0}^{m+n} B_j^{(m,n)} h_j(x) ,$$

we find

$$B_j^{(m,n)} = \sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^s \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} \begin{bmatrix} r+s \\ j \end{bmatrix} q^{\frac{1}{2}s(s-1)} . f_{m-r} g_{n-s} h'_{r+s-j} a^r b^s .$$

It follows that

$$\sum_{m,n=0}^{\infty} B_j^{(m,n)} \frac{u^m v^n}{(q)_m (q)_n} = \frac{f(u)g(v)}{(q)_j} \sum_{k=0}^{\infty} \frac{h'_k}{(q)_k} \sum_{s=0}^{k+j} (-1)^s q^{\frac{1}{2}s(s-1)} \begin{bmatrix} k+j \\ s \end{bmatrix} (au)^{k+j-s} (bv)^s$$

$$(5.17) \qquad = \frac{f(u) g(v)}{(q)_j} \sum_{k=0}^{\infty} \frac{h'_k}{(q)_k} (au)^{k+j} \left(\frac{bv}{au} \right)_{k+j}$$

$$= \frac{f(u) g(v)}{(q)_j} (au)^j \left(\frac{bv}{au} \right)_j \sum_{k=0}^{\infty} \frac{h'_k}{(q)_k} (au)^k \left(\frac{q^j bv}{au} \right)_k ,$$

where

$$(a)_k = (1-a)(1-qa) \ldots (1-q^{k-1} a) .$$

We again take $h(t) = (e(t))^{-1}$, so that $h'_k = 1$. Then the sum in (5.17) reduces to

$$\sum_{k=0}^{\infty} \frac{(au)^k}{(q)_k} \left(\frac{q^j bv}{au} \right)_k = \frac{e(au)}{e(q^j bv)} = \frac{e(au)}{(bv)_j e(bv)} .$$

We have therefore

$$(5.18) \qquad \sum_{m,n=0}^{\infty} B_j^{(m,n)} \frac{u^m v^n}{(q)_m (q)_n} = \frac{f(u) g(v) e(au)}{e(bv)} \frac{(au)^j}{(q)_j (bv)_j} \left(\frac{bv}{au} \right)_j .$$

Finally, if we multiply by $(bv)_j t^j$ and sum, we get

$$(5.19) \qquad \sum_{m,n=0}^{\infty} (bv)_j B_j^{(m,n)} \frac{u^m v^n}{(q)_m (q)_n} t^j = \frac{f(u) g(v) e(au)}{(e(bv))^2} ,$$

where in both (5.18) and (5.19) $\bar{g}_n(x)$ is defined by (5.16) and $h(t) = (e(t))^{-1}$.

REFERENCES

1. L. CARLITZ, *Eulerian numbers and polynomials*, Mathematics Magazine, vol. 33 (1959), pp. 247-260.

2. L. CARLITZ, *Some polynomials related to theta functions*, Annali di matematica pura ed applicata, vol. 41 (1955), pp. 359-373.

3. L. CARLITZ, *The product of several Hermite or Laguerre polynomials*, Monatshefte für Mathematik, vol. 66 (1962), pp. 393-396.

4. N. E. NÖRLUND, *Vorlesungen über Differenzenrechnung*, Berlín, 1924.