LINEARITY GEOMETRY. I.*

by

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0. INTRODUCTION. Although not as basic as incidence, the concept of linearity plays an important role in both classical and modern geometry. In this paper we begin the development of a geometric theory in which the basic undefined notion is that of linearity of three points. A similar study for abstract betweenness has been made by HASHIMOTO [1].**

The present study is, in a sense, part of two programs :

(1) generalization and abstraction of the concept «dimention»,

(2) the study of ordered structures (see section 2).

It is hoped that the inclusion of several unsolved problems will stimulate interest.

1. BASIC ASSUMPTIONS. Consider a ternary relation defined on an arbitrary set M. If (x, y, z) is in this relation we write xyz (to be read «x, y, z linear»), while if xyz does not subsist we write $\sim xyz$. A consideration of the ordinary linearity relation in euclidean space suggests the following assumptions :

(A) *xyz implies $xPyPzP$ for any permutation P,*

(B) *xxy for all x, y in* M.

In order to develop a significant geometrical structure in M it will be necessary to assume one or more transitivities of linearity. We shall restrict ourselves to four- and five-point transitivities. A *strong four-point transitivity* of linearity is a statement necessarily

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** Numbers in brackets refer to the bibliography.

involving four « points » (elements of M) in which two linearity relations imply a third, which is valid for euclidean linearity, and from which no relation can be omitted leaving an equivalent statement. A *weak* four-point transitivity is one in which three linearity relations imply a fourth. For example :

$$(t) \quad xyz \,.\, xyp \,.\, x \neq y \rightarrow xzp \quad \text{(strong)},$$

$$(\tau) \quad xyz \,.\, xyp \,.\, xzp \rightarrow yzp \quad \text{(weak)}.$$

Note that $x \neq y$ is required in (t) in order that the statement be valid for euclidean linearity. Note also that no four-point transitivity can have more than three linearity relations to the left of the implication sign. It is easy to see that any transitivity of five points must have three linearity relations to the left of the implication sign. Those having exactly three are called *strong*. The following theorem severely limits the choice of transitivities.

THEOREM 1. *The only four-point transitivities of linearity are* (t) *and* (τ). *Every strong five-point transitivity is equivalent to* (t), *and hence* (t) *implies every five-point transitivity.*

PROOF. Let x, y, z, p, be four points of M. Any two pairwise distinct triples selected from these points must have two points in common. In view of (A), then, we may assume the hypothesis of any strong four-point transitivity to contain $xyz \,.\, xyp$. As remarked above it is also necessary to require $x \neq y$ in order to get any meaningful statement for euclidean linearity. There are then two possible strong transitivities : (t) and $xyz \,.\, xyp \,.\, x \neq y \rightarrow yap$. But these are equivalent under the permutation (xy). Obviously any weak four-point transitivity is equivalent to (τ). Now let x, y, z, p, q, be five points in M. Of any three pairwise distinct triples selected from these points, some two of the triples must have two common elements. Therefore the hypothesis of any strong five-point transitivity may be assumed to contain $xyz \,.\, xyp$.

Eliminating statements which do not involve five points as well as etatements which are equivalent after relabelling, there remain the following three basic hypotheses :

$$T_1 : xyz \,.\, xyp \,.\, xyq,$$

$$T_2 : xyz \,.\, xyp \,.\, xzq,$$

$$T_3 : xyz \,.\, xyp \,.\, zpq.$$

These statements are invarient, by (A), under the following permutations: T_1: (xy), (zp), (zq), and (pq); T_2: (yz) (pq); T_3: (xy) and (zp). Use of these permutations serves to reduce the twenty-one meaningful implications of these hypotheses to nine. These are listed below together with those inequalities required for the implications to hold for euclidean linearity.

$$T_{11} : xyz \,.\, xyp \,.\, xyq \,.\, x \neq y \rightarrow xzp$$

$$T_{12} : xyz \,.\, xyp \,.\, xyq \,.\, x \neq y \rightarrow zpq$$

$$T_{21} : xyz \,.\, xyp \,.\, xzq \,.\, x \neq z \rightarrow xyq$$

$$T_{22} : xyz \,.\, xyp \,.\, xzq \,.\, x \neq y \,.\, x \neq z \rightarrow xpq$$

$$T_{23} : xyz \,.\, xyp \,.\, xzq \,.\, x \neq y \rightarrow yzp$$

$$T_{24} : xyz \,.\, xyp \,.\, xzq \,.\, x \neq y \,.\, x \neq z \rightarrow ypq$$

$$T_{31} : xyz \,.\, xyp \,.\, zpq \,.\, z \neq p \rightarrow xyq$$

$$T_{32} : xyz \,.\, xyp \,.\, zpq \,.\, x \neq y \rightarrow xzp$$

$$T_{33} : xyz \,.\, xyp \,.\, zpq \,.\, x \neq y \,.\, z \neq p \rightarrow xzq.$$

Now equating elements these implications follow from (B): for $x = z : T_{12}$ and T_{31} imply (t); for $y = z : T_{22}$ and T_{24} imply (t); for $x = p : T_{21}$ implies (t); for $y = q : T_{11}$ implies (t); for $z = q : T_{23}$ and T_{32} imply (t); for $p = q : T_{33}$ implies (t). Thus every transitivity implies (t). It is very easy to see that (t) implies T_{11}, T_{21}, T_{23} and T_{32}. For example $xyz \,.\, xzq \,.\, x \neq z \rightarrow yzq$ by (t), and so (t) implies T_{23}. To show that (t) implies T_{22}, T_{24}, and T_{33} requires two steps. For example $xyz \,.\, xzq \,.\, x \neq z \rightarrow xyq$ and $xyq \,.\, xyp \,.\, x \neq y \rightarrow xpq$ by (t), hence (t) implies T_{22}. For T_{12} we have $xyz \,.\, xyp \,.\, x \neq y \rightarrow yzp$ and $xyz \,.\, xyq \,.\, x \neq y \rightarrow yzq$. If $y \neq z$ then $yzp \,.\, yzq \rightarrow zpq$, while if $y = z$ then $x \neq z$ and $xzp \,.\, xzq \rightarrow zpq$. Thus (t) implies T_{12}. Finally, for T_{31}, note that if $x = y$ then xyq and if $p = q$ then xyq (since xyp). Assuming $x \neq y$, $p \neq q$ we have $xyz \,.\, xyp \rightarrow xzp$ and yzp, $zpx \,.\, zpq \,.\, z \neq p \rightarrow pxq$, $zpy \,.\, zpq \,.\, z \neq p \rightarrow ypq$, and $pqx \,.\, pqy \rightarrow xyq$. Thus (t) implies T_{31}. This completes the proof.

This theorem is analogous to the results in Part I of PITCHER and SMILEY's investigation of abstract betweenness [2]. It would be interesting to show that every strong n-point transitivity of linearity is equivalent to (t).

For the remainder of this paper we shall assume the linearity relation satisfies (A), (B) and (t).

2. Linearity in terms of other relations

(a) *Order.* Let $\leq$ be a binary relation on M. We say x and y are *comparable*, written xCy, if $x \leq y$ or $y \leq x$. x and y are *strictly comparable*, $xC'y$, if xCy and $x \neq y$. Assume

(O1) $\leq$ *is a partial ordering of* M.

(O2) $zCx \,.\, pCx \,.\, xC'y \rightarrow zCp$,

and define xyz to mean $xP \leq yP \leq zP$ or $xP = yP$ for some permutation P.

This linearity relation is easily seen to satisfy (A), (B), and (t).

(b) *Comparability.* The definition used in (a) suggests taking xCy as a primitive relation. Indeed let C be any equivalence relation and define xyz to mean $xCy \,.\, yCz$ or two of x, y, z equal. Clearly (A) and (B) are satisfied. A simple argument shows that (t) holds.

(c) *Betweenness.* Consider a ternary relation (xyz) on M which satisfies the following.

(B1) $(xyz) \rightarrow (zyx)$,

(B2) $(xyz) \,.\, (xzy) \rightarrow y = z$,

(B3) (xxy) *for all* x, y,

(B4) $(xyz) \,.\, (xpy) \rightarrow (pyz)$,

(B5) $(xyz) \,.\, (xpy) \rightarrow (xpz)$,

(B6) $(xyz) \,.\, (yzp) \,.\, y \neq z \rightarrow (xyp)$,

(B7) $(xyz) \,.\, (xyp) \,.\, x \neq y \rightarrow (xzp)$ *or* (xpz),

(B8) $(xzy) \,.\, (xpy) \rightarrow (xzp)$ *or* (xpz).

These postulates all hold for euclidean betweenness. All except (B6), (B7), and (B8) hold in a general metric space with distance $d(x, y)$, where (abc) is defined to mean $d(a, b) + d(b, c) = d(a, c)$. (These and other postulates will be discussed for various types of betweenness in an arbitrary *lattice* in a later paper.) Using the results of Huntington and Kline [3] (B1)-(B8) may be shown to be independent. If xyz is now defined to mean (xyz) *or* (xzy) *or* (yzx), then (A) and (B) follow from (B1)-(B3) and (t) follows from (B4)-(B8).

(d) *Triangle area.* (See [4]). Consider a real-valued function of three variables $a(x, y, z)$, defined on M which satisfies the following·

(A1) $a(x, y, z) \geq 0$ *for all* x, y, z,

(A2) $a(x, y, z) = a(xP, yP, zP)$ *for any permutation* P,

(A3) $a(x, x, y) = 0$ *for all* x, y,

(A4) $a(x, y, z) + a(x, y, p) + a(x, z, p) \geq a(y, z, p)$.

If xyz is now defined to mean $a(x, y, z) = 0$ then (A), (B), and (τ) are are clearly satisfied, but (t) may not hold. For example if $\text{M} = \{x, y, z, p\}$, $a(xP, zP, pP) = a(yP, zP, pP) = 1$ for all permutations P, all other values zero, then (A1)-(A4) hold but (t) fails. The direct generalization of (A4),

(A4′) $a(x, y, z) + a(x, y, p) \geq a(x, z, p)$,

does not hold for triangle area in the plane, as may be easily seen. Thus far no postulate other than the obvious one,

$$a(x, y, z) = 0 \,.\, a(x, y, p) = 0 \,.\, x \neq y \rightarrow a(x, z, p) = 0,$$

has been found which holds for triangle area in the plane and which yields (t). In searching for such a postulate it might be helpful to note that $a(x, y, z) = a(x, y, p) = 0$ implies $a(x, z, p) = a(y, z, p)$ by (A4).

3. Lines. A subset S of M is *linearly closed* if a, b in S, $a \neq b$, and abx always implies x in S. The intersection of all linearly closed sets containing a set $\{a, b, \ldots\}$ is called the linearly closed set *generated* by $\{a, b, \ldots\}$ and denoted $\text{S}(a, b, ..)$ A *straight* subset of M is one in which every triple is linear. For $a \neq b$ in M the *line* $\text{L}(a, b)$ is defined as follows.

$$\text{L}(a, b) = \{x : abx\}.$$

(By (B) $\text{L}(a, a) = \text{M}$ for any a in M.)

THEOREM 2. *Let a, b be distinct points in* M. *Then :*

(1) $L(a, b) = S(a, b)$.

(2) $L(a, b)$ *is the largest straight set containing a and b.*

(3) *if c, d are distinct points of* $L(a, b)$ *then* $L(c, d) = L(a, b)$.

PROOF. (1). a and b are in $L(a, b)$ by (B) and x in $L(a, b)$ implies abx by definition. Thus $L(a, b)$ is part of every linearly closed set containing a and b, i.e. $S(a, b) \supset L(a, b)$. On the other hand if x, y are distinct elements of $L(a, b)$ and if xyz for some z in M, then $abx \,.\, aby \,.\, xyz \,.\, a \neq b \,.\, x \neq y$ which by T_{33} gives abz, i. e. z in $L(a, b)$. Thus $L(a, b)$ is linearly closed, and so $S(a, b) \subset L(a, b)$.

(2). Let x, y, z be points of $L(a, b)$. Then $abx \,.\, aby \,.\, abz \,.\, a \neq b$, so by T_{12} xyz. Thus $L(a, b)$ is straight. Let G be any straight set containing a and b and consider any x in G. Then abx, i.e. x is in $L(a, b)$, and so $G \subset L(a, b)$.

(3). Suppose x is in $L(c, d)$. Then by T_{31} $abc \,.\, abd \,.\, cdx \,.\, c \neq d \rightarrow abx$, hence x is in $L(a, b)$. Thus $L(c, d) \subset L(a, b)$ and in the same way $L(a, b) \subset L(c, d)$. This completes the proof.

Combining (1) and (2) and reinterpreting (3) we have the COROLLARY.

(1) *A line is the only linearly closed straight set containing two given distinct points.*

(2) *Two distinct lines intersect in at most one point.*

4. PLANES. Let $\mathcal{L}$ be the set of all lines in M. (M, $\mathcal{L}$) is clearly a *partial plane* in the sense that two distinct elements of M determine a unique element of $\mathcal{L}$ and two distinct elements of $\mathcal{L}$ determine at most one element of M.

The concept of dimention may be introduced in M as follows. A subset T of M will be called *(linearly) independent* provided $p \notin S(T - \{p\})$ for all p in T. The *dimention* of a subset K of M is then given by the equation

$$1 + \dim K = \sup \{ |T| : T \subset K \text{ and } T \text{ independent} \},$$

where $|T|$ denotes the cardinality of T. It is easy to see that $\dim M \geq -1$, $\dim M = -1$ iff M is empty, $\dim M = 0$ iff $|M| = 1$, and $\dim M = 1$ iff M is a line. The planar (dimention two) case is of course more complicated.

DEFINITION. *A* semi-projective plane *is a partial plane in which, given any line* L *and point* p *not on* L, *there is at most one line containing* p *which does not intersect* L.

If dim M ≤ 1 then (M, $\mathcal{L}$) is a semi-projective plane by default. A statement equivalent to the definition is that the « parallel » relation: L || N provided L $=$ N or L does not intersect N, is an equivalence relation on L.

A discussion of B-L (BOLYAI-LOBACHEVSKY) planes may be found in [5] and references there.

THEOREM 3. *For* $a, b, c,$ *in* M *let*

$$\text{U}(a, b, c) = \{d : d = c \text{ or for all } x \; cdx \rightarrow \sim abx\}.$$

Then : (1) dim U$(a, b, c) \leq 1$ *for all,* a, b, c *if and only if* (M, $\mathcal{L}$) *is a semi-projective plane.*

(2) dim U$(a, b, c) \leq 1$ *for all* a, b, c *implies* dim M ≤ 3 ; *and* dim M $= 3$ *if and only if* (M, $\mathcal{L}$) *consists of two non-intersecting lines.*

(3) dim U$(a, b, c) = 0$ *for all* a, b, c *if and only if* (M, $\mathcal{L}$) *is either a point, a line, two intersecting lines, or a projective plane.*

(4) dim U$(a, b, c) = 1$ *for all non-linear* a, b, c *if and only if* (M, $\mathcal{L}$) *is either two non-intersecting lines or an affine plane.*

(5) dim U$(a, b, c) > 1$ *for all non-linear* $a, b, c,$ *if and only if* (M, $\mathcal{L}$) *is a* B-L *plane.*

PROOF. (1). Let c be a point not on L $=$ L(a, b) and suppose two lines L_1, L_2 through c do not intersect L. Then $\text{L}_1 \cup \text{L}_2 \subset \text{U}(a, b, c)$, so dim U$(a, b, c) \geq 2$. Conversely if (M, $\mathcal{L}$) is semi-projective then for any a, b, c U(a, b, c) is either $\{c\}$ or a line (or empty), and so dim U$(a, b, c) \leq 1$.

(2). Suppose M contains five independent points a, b, c d, e. If for some x abx and cdx then $d \in S(a, b, c)$, contradicting independence. Hence d is in U(a, b, c) and similarly so is e. c is in U(a, b, c) by definition. But $\sim cde$, whence dim U$(a, b, c) \geq 2$, a contradiction, which implies dim M ≤ 3. Now suppose dim M $= 3$ and a, b, c, d are independent. Then L(a, b) does not intersect L(c, d). Suppose there were a point e not on L(a, b) or L(c, d). Then L(c, e) and L(d, e) would meet L(a, b) in distinct points, say f, g respectively. Then

$c, f \in S(a, b, c)$, so $e \in S(a, b, c)$, and $e, g \in S(a, b, c)$ implies $d \in S(a, b, c)$ contradicting independence. Thus M consists of L(a, b) and L(c, d).

It should now be clear how (3), (4), and (5) are proved.

Combining Theorem 3 part (3) with section 2 part (c) we obtain a characterization of projective planes in terms of betweenness, which has been requested by several mathematicians.

THEOREM 4. *Let* M *be any set on which there is defined a ternary relation* (xyz) *satisfying* (B1)-(B8),

(B9) *Given* a, b, c, d *in* M *there exists* x *in* M *such that*

$$(abx \text{ or } axb \text{ or } xab) \text{ and } (cdx \text{ or } cxd \text{ or } xcd),$$

(B10) *There exist* a, b, c, d *in* M, *pairwise distinct, such that* $\sim (xyz)$ *for all distinct* x, y, z *in* $\{a, b, c, d\}$.

Then (M, $\mathcal{L}$) *is a projective plane with lines defined by*

$$L(a, b) = \{x : abx \text{ or } axb \text{ or } xab\}, \text{ for } a \neq b.$$

PROOF. (B9) gives the condition of (3), Theorem 3, while (B10) requires dim M ≥ 2 and also eliminates the case of exactly two intersecting lines.

A similar characterization of affine planes exists.

Another approach to these «generalized planes» makes use of *k-sets* of lines (*k-spaces* in [6]). Consider three distinct points a, b, c in M. Let $\mathcal{L}_0 = \{a, b, c\}$ and $\mathcal{L}_k = \{L(x, y) : x \neq y, x, y \in \bigcup \mathcal{L}_{k-1}\}$, $k = 1, 2, \ldots$ If abc then $\mathcal{L}_1 = \mathcal{L}_2 = \ldots$ If $(M, \mathcal{L})$ is semi-projective and $\sim abc$ then $\mathcal{L} = \mathcal{L}_2$. The general structure of k-sets will be examined in a later paper.

5. PLANES IN SPACE. Throughout this section we assume each line in M contains at least three points. If planes are to be indentified with the sets S(a, b, c), where $\sim abc$, the fundamental requirement will be

(D) $e, f, g \in$ S(a, b, c), $\sim efg \rightarrow$ S$(e, f, g) =$ S(a, b, c).

This is implied by the « Pasch's axiom »

(E) *In* S(a, b, c) *if a line meets one side of a triangle it meets another side.*

For if (E) is assumed and $a \notin$ L(e, f) we may choose $h \in$ L(e, f) distinct from e, f, and a. Then the line L(a, h) intersects the triangle

(e, f, g) in h and hence in another point k which must be distinct from h. Therefore since $h, k \in S(e, f, g)$ it follows that in any case $a \in S(e, f, g)$, and in the same way that $b, c \in S(e, f, g)$. Thus $S(a, b, c) \subset S(e, f, g)$ and the sets must be equal. (D) does not imply (E) for example in the finite non-homogeneous B-L plane of [5], where the line AKM meets exactly one side of the triangle (F, E, J.). Indeed (E) fails in any B-L plane, while (E) holds in any semi-projective plane. (What can be said about the largest class of designs in which (E) holds)

THEOREM 5. *Suppose* $\dim M > 2$, (D) *holds in* M, *and every line in* M *meets every plane. Then* M *is a projective* 3-*space.*

PROOF. Let $P_1 = S(a, b, c)$, $P_2 = S(d, e, f)$ be distinct planes. We may assume $a \notin P_2$. Then $L(a, b)$ and $L(a, c)$ do not lie in P_2 and hence intersect P_2 in points, say h and k. $h \neq k$, since $\sim abc$, so $L(h, k) \subset P_1 \cap P_2$. If $P_1 \cap P_2$ contains a point p not on $L(h, k)$, it contains the plane $P = S(p, L(h, k))$, and $P_1 = P = P_2$. Thus two distinct planes meet in a line. Now let L_1, L_2 be distinct lines in a plane P. Since $\dim M > 2$ there is a point a of M which does not lie in P. Then $P_1 = S(a, L_1)$ and $P_2 = S(a, L_2)$ are distinct planes, since $P_1 = P_2$ would imply $P = S(L_1, L_2) \subset P_1$, $P = P_1$, and $a \in P$. Let L be the line of intersection of P_1 and P_2. L does not lie in P since $a \in L$, so $L \cap P$ is a single point, say p. Then $p \in P_1 \cap P = L_1$ and $p \in P_2 \cap P = L_2$, so $p \in L_1 \cap L_2$, and in fact $\{p\} = L_1 \cap L_2$. Thus P is a projective plane, and so M is a projective 3-space.

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