A COMPLEX INVERSION THEOREM FOR A GENERALIZATION OF LAPLACE TRANSFORM

by

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1. Introduction.

If we apply Kober's Operators (1) of Fractional Integration to $x^{\beta} e^{-x}$ we get two kernels. These kernels may be used to define functional transformations. In a recent paper (2) I have discussed the convergence properties and some inversion and representation theorems (3) of the functional transformation (4).

$$(1.1) \quad F(x) = \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)} \int_0^{\infty} (x\,y)^{\beta}\, F(\beta+\eta+1,\ \alpha+\beta+\eta+1, -xy)\, f(y)\, dy$$

$$= A \int_0^{\infty} (x\,y)^{\beta}\, F(x, y)\, f(y)\, dy$$

where for convenience we write A for $\dfrac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+\alpha\, 1)}$ and $F(x, y)$ for ${}_1F_1(\beta+\eta+1,\ \alpha+\beta+\eta+1,\ -x\,y)$

For $\alpha = \beta = 0$ (1.1) reduces to the well known Laplace Transform.

$$(1.2) \qquad F(x) = \int_0^{\infty} e^{-xy}\, f(y)\, dy$$

In the present paper I am giving an inversion theorem for (1.1).

2. Complex inversion theorem

We now prove

THEOREM 2.1. If $F(x)$ is given by the convergent integral (1.1) then

$$\frac{1}{2}\left[f(y+)+f(y-)\right] \tag{2.2}$$

$$=\frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{c-i\tau}^{c+i\tau}\frac{\Gamma(\alpha+\eta+s)}{\Gamma(\beta+1-s)}\times\frac{y^{-s}}{\Gamma(\eta+s)}\varphi(s)\,ds$$

where

$$\varphi(s)=\int_0^\infty x^{-s} f(x)\,dx \tag{2.3}$$

provided that

(*i*) $t^{c-s} f(t) \epsilon L(0,\infty)$

(*ii*) $t^{-s} F(t)\ \varepsilon\ L(0,\infty)\ (s=c+i\tau,\ -\infty<\tau<\infty.$

(*iii*) $f(t)$ is of bounded variation in the neighbourhood of the point

$$t=y\ \ (y>0).$$

(*iv*) $f(t)=0\ (t^{\varrho-\beta})\ Re\ \varrho>0,\ (t\to 0)$

(*v*) $=(e^{-t\nu}),\ Re\ \nu>0,\ (t\to\infty)\ 0<Re\,(\beta-s)<Re\,(\beta+\eta)$

If $f(t)$ is continuous at $t=y$ we have

$$f(t)=\frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{c-i\tau}^{c+i\tau}\frac{\Gamma(\alpha+\eta+s)}{\Gamma(\beta+1-s)\,\Gamma(\eta+s)}y^{-s}\varphi(s)\,ds$$

PROOF: We have

$$\int_0^\infty x^{-s} F(x)\,dx = A\int_0^\infty x^{-s+\beta}\,dx\int_0^\infty y^{\beta} F(x,y) f(y)\,dy$$

$$=A\int_0^\infty y^{\beta} f(y)\,dy\int_0^\infty x^{-s+\beta} F(x,y)\,dx$$

provided that we can justify the change in the order of integration. Hence (5, page 48).

$$\int_0^\infty x^{-s} F(x)\,dx=\frac{\Gamma(\eta+s)\,\Gamma(\beta-s+1)}{\Gamma(\alpha+\eta+s)}\int_0^\infty y^{s-1} f(y)\,dy$$

Where $0 < Re\,(\beta - s) < Re\,(\beta + \eta + 1)$

If we apply MELLIN'S Inversion formula (6) to the integral

$$\int_0^{\infty} y^{s-1} f(y)\, dy = \frac{\Gamma(\alpha + \eta + s)}{\Gamma(\beta - s + 1)\,\Gamma(\eta + s)}\, \Phi(s)$$

We have

$$\frac{1}{2}\left[f(y+) + f(y-) \right]$$

$$= \frac{1}{2\pi i} \lim_{\tau \to \infty} \int_{c-i\tau}^{c+i\tau} \frac{\Gamma(\alpha + \eta + s)}{\Gamma(\beta - s + 1)\,\Gamma(\eta + s)}\, y^{-s} \varphi(s)\, ds$$

Where

$$\Phi(s) = \int_0^{\infty} x^{-s} F(x)\, dx$$

provided that,

(*i*) $t^{c-1} f(t)\ \varepsilon\ L\,(0, \infty)$

(*ii*) $t^{-s} F(t)\ \varepsilon\ L\,(0, \infty)$

(*iii*) $f(t)$ is of bounded variation in the neighbourhood of the point $t = y$ $(y > 0)$.

Now we shall justify the change in the order of integration.

We easily see that

$x^{-s+\beta} \int_0^{\epsilon} y^{\beta} F(x, y) f(y)\, dy$ is uniformly convergent in $x \geq 0$ provided that $Re\,(\varrho + 1) > 0$ $Re\,(\beta - s + \varrho + 1) \geq 0$ since (5 page 60)

$${}_1F_1\,(a, b, -t) = 0\,(1)\ (t \to 0).$$

Also, since

$${}_1F_1\,(a, b, -t) \sim t^{-a}\ (t \to \infty)\ Re\,(a) > 0$$

we have,

$$y^{\beta} f(y) \int_0^{\infty} x^{-s+\beta} f(x, y)\, dx$$

is uniformly convergent in $y \geq 0$ provided that $Re\ (\beta - s + 1) > 0$, $Re\ (\eta + s) > 0$, $Re\ \varrho \geq 0$

Again it is clear that the integral

$$\int_{y}^{\infty} | y^{\beta} f(y) | \, dy \int_{y'}^{\infty} | x^{-s+\beta} F(x, y) | \, dx$$

where y and y' are large, does not exceed a constant multiple of

$$\int_{y}^{\infty} | e^{-y\nu} | \, dy \int_{y'}^{\infty} | x^{-(s+\eta+1)} | \, dx$$

by our hypotheses on $f(t)$, which tends to zero if $Re\ (\nu) > 0$ and $Re\ (s + \eta) > 0$

3. A particular case

If $\alpha = \beta = 0$ we have the following theorem.

THEOREM 3.1. If

$$F(x) = \int_{0}^{\infty} e^{-xy} f(y) \, dy \quad \text{converges}$$

then

$$(3.2) \quad \frac{1}{2}\left[f(y + 1 + f(y -) \right] = \frac{1}{2\pi i} \lim_{\tau \to \infty} \int_{c-i\tau}^{c+i\tau} \frac{1}{\Gamma(1-s)} y^{-s} \Phi_1(s) \, ds$$

where

$$\Phi_1(s) = \int_{0}^{\infty} x^{-s} F(x) \, ds$$

provided that conditions (i) to (iv) of theorem 2.1 hold and

$$0 < Re\ (-s) < 1.$$

4. Examples

We now give two examples to verify theorem 2.1 and theorem 3.1.

EXAMPLE 4.1. Let us take

$$f(y) = y^{\varrho} J_{\nu}(y^{\frac{1}{2}})$$

then

$$F(x) = A \int_0^{\infty} (x\,y)^{\beta} F(x, y)\, y^{\varrho} J_{\nu}(y^{\frac{1}{2}})\, dy$$

$$= A\,\frac{1}{A}\,2^{2\varrho-2\beta-1}\,x^{\beta}\,G\left(\frac{1}{4x}\,\middle|\,\begin{matrix}1,\ \alpha+\beta+\eta+1,\\ 1+\varrho+\beta+\frac{1}{2}\nu,\ \beta+\eta+1,\ 1+\varrho+\beta-\nu/2\end{matrix}\right).$$

provided that

$$-1 - Re(\nu) < 2\,Re\,\varrho < \tfrac{1}{2} + 2\,Re(\beta+\eta+1)$$

since (7, page 88 (*b*))

$$\int_0^{\infty} x^{2\varrho-\frac{1}{2}} F(\alpha, \beta, -\lambda x^2)\, J_{\nu}(x\,y)\,(xy)^{\frac{1}{2}}\,dx$$

$$= \frac{2^{2\varrho}\,\Gamma(\beta)}{\Gamma(\alpha)\,y^{2\varrho+1/2}}\,G^{2,\ 1}_{2,\ 3}\left(\frac{y^2}{4\lambda}\,\middle|\,\begin{matrix}1,\ \beta\\ \frac{1}{2}+\varrho+\frac{\nu}{2},\ \alpha,\ \frac{1}{2}+\varrho-\frac{\nu}{2}\end{matrix}\right)$$

provided that

$$-1 - Re(\nu) < 2\ Re(\varrho) < \tfrac{1}{2} + 2\ Re(\alpha),\ \ Re(\lambda) > 0,\ \ Re(y) > 0.$$

Therefore $\Phi(s) = \int_0^{\infty} x^{-s}\, F(x)\, d\,x$

$$= 2^{2\varrho-\beta-1}\int_0^{\infty} x^{\beta-s}\, G^{2\,1}_{2\,3}\left(\frac{1}{4\,x}\,\middle|\,\begin{matrix}1,\ \alpha+\beta+\eta+1\\ 1+\varrho+\beta+\nu/2,\ \beta+\eta+1,\ 1+\varrho+\beta-\nu/2)\end{matrix}\right) dx$$

$$= 2^{2\varrho-2s}\int_0^{\infty} t^{s-\beta-2}\, G^{2\,1}_{2\,3}\left(t\,\middle|\,\begin{matrix}1,\ \alpha+\beta+\eta+1,\\ 1+\varrho+\beta+\nu/2,\ \beta+\eta+1,\ 1+\varrho+\beta-\nu/2\end{matrix}\right) dt$$

$$= \frac{2^{2\varrho+2s}\,\Gamma(1+\varrho+\beta+\nu/2+s-\beta-1)\,\Gamma(\beta+\eta+1+s-\beta-1)\,\Gamma(\beta-s+1)}{\Gamma(1-1-\varrho-\beta+\nu/2-s+\beta+1)\,\Gamma(\alpha+\beta+\eta+1+s-\beta-1).}$$

$$= \frac{2^{2\varrho+2s}\ \Gamma(\varrho+\nu/2+s)\,\Gamma(\eta+s)\,\Gamma(s-\beta)}{\Gamma(\nu/2-\varrho-s+1)\,\Gamma(\alpha+\eta+s)}$$

provided that (8, page 337 (14),),

$$-\min Re\,(1+\varrho+\beta+\nu/2,\ \beta+\eta+1) < 1-\max\,(1,\ \alpha+\beta+\eta+1).$$

Therefore

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\alpha+\eta+s)}{\Gamma(\beta+1-s)\,\Gamma(\eta+s)}\,y^{-s}\,\varphi(s)\,sd$$

$$=\frac{1}{2\pi i}\,2^{2\varrho}\int_{c-i\infty}^{c+i\infty}\frac{2^{2s}\,\Gamma(\varrho+\nu/2+s)}{\Gamma(\nu/2-\varrho-s+1)}\,y^{-s}\,ds \equiv I_1 = y^{\varrho}\,J_\nu\,(y^{\frac{1}{2}})$$

since (9, page 219 (44))

$$2^{2\varrho}\,G^{\,1,\ 0}_{\,0,\ 2}\left(\frac{t}{4}\middle|\,\nu/2+\varrho,\ \ \varrho-\nu/2\right) = I_1 = y^{\varrho}\,J_\nu\,(y^{\frac{1}{2}})$$

using the definition of G function.

Example 4.2

Let $f(y) = y^{\varrho/2}\,y_\nu\,(y^{\frac{1}{2}})$

then $F(x) = \int_0^\infty e^{-xy}\,f(y)\,dy$

$$=\int_0^\infty e^{-xy}\,y^{\varrho/2}\,y_\nu\,(y^{\frac{1}{2}})\,dy$$

$$=\frac{1}{x^{\partial+\frac{1}{2}}}\,G^{\,3,\ 1}_{\,1,\ 4}\left(\frac{1}{4x}\middle|\begin{matrix}-\varrho/2,\ -\varrho/2-1,\ -\frac{1}{2}-\nu/2\\ \nu/2,\ -\nu/2,\ -\varrho/2-1,\ -\frac{1}{2}-\nu/2\end{matrix}\right)$$

provided that $Re\,x > 0$, $Re\,(\varrho+2) > |\,Re\,(\nu)\,|$ $3/2 > Re\,(\varrho+2)$
since (7, page 119)

$$\int_0^\infty \lambda^{\sigma^3/2}\,{}_1F_1\,(0,\ 0,\ -\lambda\,x^2)\,y_\nu\,(x\,y)\,(x\,y)^{\frac{1}{2}}\,dx$$

$$=\frac{y^{1/2}}{2\lambda^{\sigma/2}}\,G^{\,3\ 1}_{\,3\ 4}\left(\frac{y^2}{4\lambda}\middle|\begin{matrix}1-\sigma/2,\ -\sigma/2-1,\ -\frac{1}{2}-\nu/2\\ \nu/2,\ -\nu/2,\ -\sigma/2-1,\ -\frac{1}{2}-\nu/2\end{matrix}\right)$$

provided that $Re\,(\lambda) > 0$, $Re\,(\sigma) > Re\,|\,(\nu)\,|$, $0 > \frac{1}{2}\,Re\,\sigma - \frac{3}{4}$

Therefore $\Phi_1(s) = \int_0^\infty x^{-s} F(x)\, dx$

$$= \int_0^\infty \varkappa^{-s-\varrho/2-1}\, G^{3\,1}_{3\,4}\left(\frac{1}{4\varkappa}\middle|\begin{matrix} -\varrho/2,\ -\varrho/2-1,\ -\tfrac{1}{2}-\nu/2) \\ \nu/2,\ -\nu/2,\ -\varrho/2-1,\ \tfrac{1}{2}-\nu/2 \end{matrix}\right) d\varkappa$$

$$= \frac{2^{2s+\varrho}\, \Gamma(\nu/2+\varrho/2+s)\, \Gamma(-\nu/2+\varrho/2+s)\, \Gamma(1-s)}{\Gamma(\tfrac{3}{2}+\nu/2-s-\varrho/2)\, \Gamma(-\tfrac{1}{2}+\nu/2+\varrho/2+s)}$$

On using the result (14) of page 337 (8), provided that,

$$-\min Re(\nu/2, -\nu/2, -\nu/2-1) < 1 - \max Re(-\varrho/2, -\varrho/2-1, -\tfrac{1}{2}-\nu/2)$$

Hence

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{\Gamma(1-s)} y^{-s}\, \Phi_1(s)\, ds = y^{\varrho/2}\, \gamma_\nu(y^{\frac{1}{2}})$$

since (2, page 219 (46),),

$$y^{\varrho/2}\, \gamma_\nu(y^{\frac{1}{2}})$$

$$= 2^\varrho\, G^{2,\ 0}_{1,\ 3}\left(\frac{y}{4}\middle|\begin{matrix} \varrho/2-\nu/2-\tfrac{1}{2} \\ \varrho/2-\nu/2,\ \varrho/2+\nu/2,\ \varrho/2-\nu/2-\tfrac{1}{2} \end{matrix}\right)$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2^{2s+\varrho}\, \Gamma(\nu/2+\varrho/2+s)\, \Gamma(-\nu/2+\varrho/2+s)\, y^{-s}}{\Gamma(\tfrac{3}{2}+\nu/2-s-\varrho/2)\, \Gamma(-\tfrac{1}{2}+\nu/2+\varrho/2+s)}\, ds$$

by the definition of G function.

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