ON PROXIMATE ORDER (R) OF ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES (II)

by

PAWAN KUMAR KAMTHAN
Birla College, Pilani, Raj., INDIA.

1. Let $F(s) = \sum_{n=1}^{\infty} A_n e^{s\lambda_n}$ $(s = \sigma + it;\ 0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n;$ $\overline{\lim_{n\to\infty}}\ n/\lambda_n = D' < \infty$ $\underline{\lim_{n\to\infty}}\ (\lambda_n - \lambda_{n-1}) = h > 0;\ hD' \leq 1)$ be an entire function of order (R) ϱ $(0 < \varrho < \infty)$ represented by Dirichlet series. Sunyer i Balaguer [3] has introduced the notion of proximate order (R) $\varrho(\sigma)$: according to him it is a function of σ to satisfy the following conditions:

(i) $\lim_{\sigma\to\infty} \varrho(\sigma) = \varrho$

(ii) $\lim_{\sigma\to\infty} \sigma\varrho'(\sigma) = 0$

(iii) $\overline{\lim_{\sigma\to\infty}} \dfrac{\log M(\sigma)}{e^{\sigma\varrho(\sigma)}} = 1;$ $\qquad M(\sigma) = \sup_{-\infty<t<\infty} |F(\sigma + it)|.$

Although $\varrho(\sigma)$ satisfies (iii), it will, however, be more convenient, in comparing $\log M(\sigma)$ with $\lambda_{\nu(\sigma)}$ and so on, to write

$$\overline{\lim_{\sigma\to\infty}} \frac{\log M(\sigma)}{e^{\sigma\varrho(\sigma)}} = A.$$

2. Our aim in this note is to establish certain results involving the growth of $\log M(\sigma)$ with $\lambda_{\nu(\sigma)}$ and so on. First of all we establish some general results.

3. Define:

$$A(\sigma) = \exp \int_0^{\sigma} \varrho(t)\ dt; \quad B(\sigma) = \int_0^{\sigma} f(t)\, \varrho(t)\ dt,$$

where $f(t)$ is a non-decreasing function, at least for $t \geq t_0$. Also let

$$\varlimsup_{\sigma\to\infty} \frac{f(\sigma)}{A(\sigma)} = \frac{C}{D}\,;\quad \varliminf_{\sigma\to\infty} \frac{B(\sigma)}{A(\sigma)} = \frac{E}{F}\,;$$

4. THEOREM : (i)

(4.1) $eE \geq Ce^{D/C}$; (4.2) $F \geq D$.

(4.3) $E \leq C$; (4.4) $F \leq D(1 + \log (C/D))$.

(ii) If $0 < F \leq E < \infty$, then $0 < D \leq C < \infty$ and conversely.

(iii) If $C = D$ then $E = F$ and conversely.

(iv) If (ii) holds, then

$$e^{-\xi} < \varlimsup_{\sigma\to\infty} \frac{B(\sigma)}{f(\sigma)} < e^{\xi},$$

where ξ is the root of the equation $eEx + eE - Fe^{x} = 0$, which lies in the interval $(1, \infty)$.

5. *Preliminary lemmas :*

Lemma 1 : $e^{\sigma\varrho(\sigma)}$ *is an increasing function of* σ *for* $\sigma > \sigma_0$.

For,

$$(e^{\sigma\varrho(\sigma)})' = e^{\sigma\varrho(\sigma)}(\sigma\varrho'(\sigma) + \varrho(\sigma)) > (\varrho - \epsilon)e^{\sigma\varrho(\sigma)},$$

for $\sigma > \sigma_0$; since by (i) and (ii) of §1, we have respectively

$$\sigma\varrho'(\sigma) > -\frac{\epsilon}{2}\,;\ \varrho(\sigma) > \varrho - \frac{\epsilon}{2} \text{ for } \sigma > \sigma_0.$$

Therefore, for $\sigma > \sigma_0$, $(e^{\sigma\varrho(\sigma)})' > 0$ and so the lemmas follows.

Lemma 2 : *For* $0 < k < k_1$, $\exp \{(\sigma + k)\varrho(\sigma + k) - \sigma\varrho(\sigma)\} \to \exp (\varrho k)$ *uniformly as* $\sigma \to \infty$.

For let $\zeta = \max\ \varrho'(x)$ for $\sigma \leq x \leq k_1 + \sigma$. Then

$$\frac{\exp \{(\sigma+k)\,\varrho\,(\sigma+k)\}}{\exp.\ \{\sigma\varrho(\sigma)\}} = \exp\{k\varrho(\sigma + k)\}\exp\left\{\frac{\varrho(\sigma + k) - \varrho(\sigma)}{k}k\sigma\right\}$$

$$= \exp\{k\varrho(\sigma + k)\}\ \exp\ (k\sigma\xi),$$

where $0 < \xi < \zeta$, and so the lemma follows from (i) and (ii) of §1.

Lemma 3: *This is :*

$$\frac{A(\sigma + k)}{A(\sigma)} \to e^{\varrho k}$$

uniformly as $\sigma \to \infty$.

For

$$\log \left\{\frac{A(\sigma+k)}{A(\sigma)}\right\} = \int_{\sigma}^{\sigma+k} \varrho(t)\,\mathrm{dt} = t\varrho(t)\Big|_{\sigma}^{\sigma+k} - \int_{\sigma}^{\sigma+k} t\varrho'(t)\,\mathrm{dt}.$$

But $\left|\int_{\sigma}^{\sigma+k} t\varrho'(t)\,\mathrm{dt}\right| < k\varepsilon$ for $\sigma \geq \sigma_0$ and so the result follows from lemma 2.

6. *Proof of the theorem :* Suppose $\mu \geq 0$. Then

$$B(\sigma+\mu) = \int_0^{\sigma} \frac{f(t)}{A(t)} A'(t)\ \mathrm{dt} + \int_{\sigma}^{\sigma+\mu} \frac{f(t)}{A(t)} A'(t)\ \mathrm{dt}$$

$$\therefore \frac{B(\sigma+\mu)}{A(\sigma+\mu)} \geq \frac{1}{A(\sigma+\mu)} \int_0^{\sigma} \frac{f(t)}{A(t)} A'(t)\ \mathrm{dt} + \frac{f(\sigma)}{A(\sigma+\mu)} \int_{\sigma}^{\sigma+\mu} \frac{A'(t)}{A(t)}\ \mathrm{dt}$$

$$= P + Q.$$

Now

$$\varliminf_{\sigma\to\infty} P = e^{-\varrho\mu} \varliminf_{\sigma\to\infty} \frac{\int_0^{\sigma} \frac{f(t)}{A(t)} A'(t)\ \mathrm{dt}}{\int_0^{\sigma} A'(t)\ \mathrm{dt}} \geq e^{-\varrho\mu} \varliminf_{\sigma\to\infty} \frac{f(\sigma)}{A(\sigma)} = De^{-\varrho\mu}.$$

$$\varlimsup_{\sigma\to\infty} Q = e^{-\varrho\mu} \varlimsup_{\sigma\to\infty} \frac{f(\sigma)}{A(\sigma)} \log \left\{\frac{A(\sigma+\mu)}{A(\sigma)}\right\} = \varrho\mu\, Ce^{-\varrho\mu}.$$

Therefore

(6.1) $$E = \varlimsup_{\sigma\to\infty} \frac{B(\sigma)}{A(\sigma)} \geq \varliminf_{\sigma\to\infty} P + \varlimsup_{\sigma\to\infty} Q \geq De^{-\varrho\mu} + \varrho\mu\, Ce^{-\varrho\mu}.$$

Similarly

(6.2) $$F = \varliminf_{\sigma\to\infty} \frac{B(\sigma)}{A(\sigma)} \leq Ce^{-\varrho\mu} + \varrho\mu D.$$

We also have, following (6.1) and (6.2),

(6.3) $$F \geq D(e^{-\varrho\mu} + \varrho\mu\, e^{-\varrho\mu}).$$

(6.4) $$E \leq C\ (e^{-\varrho\mu} + \varrho\mu).$$

Now the maximum and minimum of $(D + \varrho\mu C)e^{-\varrho\mu}$ and $Ce^{-\varrho\mu} + \varrho\mu D$ respectively occur at $\mu = (C\text{-}D)/\varrho C$ and $\mu = \frac{1}{\varrho} \log\ (C/D)$; and so subtituting these values of μ in (6.1) and (6.2) we get (4.1) and (4.4). Similarly subtituting $\mu = 0$ in (6.3) and (6.4) respectively, we get (4.2) and (4.3). This proves (i).

(ii). Now let $0 < F \leq E < \infty$. Then from (6.1) $C < \infty$. We say $D \neq 0$, for if it is then from (6.2) $F = 0$ and so a contradiction. Hence $D > 0$. Thus $0 < D \leq C < \infty$. Next suppose $0 < D \leq C < \infty$; and we have then from (6.4) and (6.3) $E < \infty$ and $F > 0$. This establishes (ii).

(iii) Let $\mu = 0$. Then from (6.1), (6.4) and (6.2), (6.3) we have

$$D \leq E \leq C\,;\ D \leq F \leq C.$$

Suppose first that $D = C$, then the preceding inequalities yield $E = F = C$. Next suppose $E = F$, then we show that $D = C$. To do that let η be an arbitrarily chosen positive number. Then

$$\begin{aligned}(1 + o(1))\varrho\eta f(\sigma) &= f(\sigma)\int_{\sigma-\eta}^{\sigma} \varrho(t)\ dt \\ &\geq \int_{\sigma-\eta}^{\sigma} f(t)\,\varrho(t)\ dt \\ &= B(\sigma) - B(\sigma-\eta) \\ &= A(\sigma)\ (1 + o(1))\ E - (1 + o\,(1))\ EA(\sigma-\eta) \\ &= (1 + o(1))(1 - e^{-\varrho\eta})\ E\ A(\sigma) \\ &= (1 + o(1))\ \ (\varrho\eta + 0(\eta^2))EA(\sigma)\end{aligned}$$

and since η is arbitrary, we find that

$$\lim_{\sigma\to\infty} \frac{f(\sigma)}{A(\sigma)} \geq E.$$

Similarly by considering the expression $f(\sigma)\int_{\sigma}^{\sigma+\eta} \varrho(t)\ dt$ we can show that

$$\lim_{\sigma\to\infty} \frac{f(\sigma)}{A(\sigma)} \leq E,$$

and thus (iii) is completely established.

(iv). We have from (4.1) $C < eE$. Hence from (6.2)

$$e^{\varrho\mu}\ F < eE + \varrho\mu\ De^{\varrho\mu}.$$

Consider now the equation

$$Eex = Fe^{x} - eE \qquad (0 < F \leq E < \infty)$$

It has one and only one root in the interval $(1,\infty)$. Let it be ξ. So putting $\varrho\mu = \xi$, we have

$$e^{\xi}F - Ee < \xi De^{\xi},$$

or, $$eE\xi < \xi De^{\xi}, \quad \text{or} \quad D > \frac{eE}{e^{\xi}}.$$

Hence

$$\frac{1}{e^{\xi}} < \frac{D}{C} \leq \overline{\lim_{\sigma\to\infty}} \frac{B(\sigma)}{f(\sigma)} \leq \frac{C}{D} < \frac{Eee^{\xi}}{eE} = e^{\xi}.$$

This furnishes the proof of (iv).

Applications: Let $f(\sigma) = \lambda_{\nu(\sigma)}$; $\dfrac{B(\sigma)}{\varrho} \sim \log \mu(\sigma) \sim \log M(\sigma)$ and so the above theorem includes the results of Srivastava ([2], p. 137) and Rahman ([1], p. 173).

REFERENCES

1. RAHMAN, Q. I. — *A note on entire functions (defined by Dirichlet series) of perfectly regular growth ;* Quart. J. Math. 6 (1955), 173-175.

2. SRIVASTAVA, K. N. — *On the maximum term of an entire Dirichlet series ;* Pro. Nat. Aca. Scs., (Allha.), India, 27 (1958), 134-146.

3. SUNYER I BALAGUER. — *Sobra la distribución de los valores de una función entera representada por una serie de Dirichlet lagunar;* Rev. Aca. Ciencias, Zaragoza, 5 (1950), 25-49.