THE UNIQUENESS PROBLEM IN THE THEORY OF NUMERICAL DIVERGENT SERIES AND FORMAL LAWS OF CALCULUS II (*)

by

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§ 2. Identity of an algorithm with those of EULER and BOREL.

1. *Definition of convergence and addition algorithms through the formal laws of calculus.*

Let

$\mathbf{s}_0$	$\mathbf{S}_0$
be the set of s_n numerical convergent sequences $\lim\limits_{n\to\infty} s_n < \infty$;	be the set of the numerical convergent series $\sum\limits_{n=0}^{\infty} a_n < \infty$;

and

λ_0	σ_0

the isomorphism existing between

$\mathbf{s}_0$	$\mathbf{S}_0$

and the complex field $\mathbf{C} = \mathbf{R}^2$ with respect to the formal laws of calculs used in the elemental theory of

convergent sequences	convergent series

The problem of

the generalized limits	addition of divergent series

should be attempted in our meaning in this other way [21, no, 1, 2 1, Chap, V, 339].

Problem A. To extend the isomorphism

λ_0	σ_0

by means of another isomorphism

λ	σ

between a set

$\mathbf{s}_\lambda \supset \mathbf{s}_0$ ($\mathbf{s}_\lambda \neq \mathbf{s}_0$)	$\mathbf{S}_\sigma \supset \mathbf{S}_0$ ($\mathbf{S}_\sigma \neq \mathbf{S}_0$)

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and the same complex plan $\mathfrak{C}$ with respect to certain formal laws, for instance, the laws

a, b, c, d, e and f | a′, b′, c′, d′ or d'_1, e′ and f′

stated below in the weak form that will be used.

Def. 1. We will call *permanent method* of

convergence | addition

or more briefly, *algorithm* of

convergence | addition

any solution

λ | σ

to problem A with respect to an system of formal laws.

The algorithm will be *distributive* when, among these laws, there exists the

a and b | a′ and b′

Def. 2. The

sequences $\{s_n\} \in s_\lambda$ | series $\sum_{n=0}^{n=\infty} a_n \in S_\sigma$

are called

λ-convergent ; | σ-summable ;

and the number

l | s

homologous in the isomorphism is called

λ-limit | σ-sum

of the

sequence $\{s_n\}$ | serie $\sum_{n=0}^{\infty} a_n$

and is written

$\lambda\text{-lim } s_n = l$ | $\sigma \sum_{n=0}^{\infty} a_n = s$

Def. 3. We say that an algorithm

λ | σ

s (strictly) *weaker* than another

λ' | σ'

or that

λ' | σ'

is (strictly) *stronger* than

λ | σ

sometimes written

$\lambda \subset \lambda' \; (\lambda \subset \lambda',\ \lambda \neq \lambda')$	$\sigma \subset \sigma' \quad (\sigma \subset \sigma',\ \sigma \neq \sigma')$

when it is

$s_\lambda \subset s_{\lambda'} \; (s_\lambda \subset s_{\lambda'},\ s_\lambda \neq s_{\lambda'})$	$S_\sigma \subset S_{\sigma'} \; (S_\sigma \subset S_{\sigma'},\ S_\sigma \neq S_{\sigma'})$

Prop. 1. (Transitive property). *If*

$\lambda \subset \lambda'$ and $\lambda' \subset \lambda''$, $\lambda \subset \lambda''$	$\sigma \subset \sigma'$ and $\sigma' \subset \sigma''$, $\sigma \subset \sigma''$

Proof. Evidently, if

$s_\lambda \subset s_{\lambda'}$ and $s_{\lambda'} \subset s_{\lambda''}$, $s_\lambda \subset s_{\lambda''}$	$S_\sigma \subset S_{\sigma'}$ and $S_{\sigma'} \subset S_{\sigma''}$, $S_\sigma \subset S_{\sigma''}$

Def. 4. If

$\{\lambda_\alpha\}$	$\{\sigma_\alpha\}$

is an algorithms family of

convergence	addition

with a variable parameter α within a set F, we will say that the *set or union* of these algorithms

$\cup\,(\lambda_\alpha,\ \alpha \in F)$	$\cup\,(\sigma_\alpha,\ \alpha \in F)$

is *(strictly) weaker* than an algorithm

λ	σ

or that the latter is *(strictly) stronger* than the said set of algorithms, and may write

$\cup(\lambda_\alpha,\alpha\in F)\subset\lambda,(\cup(\lambda_\alpha,\alpha\in F)\subset\lambda,\cup(\lambda_\alpha,\alpha\in F)\neq\lambda)$	$\cup(\sigma_\alpha,\alpha\in F)\subset\sigma,(\cup(\sigma_\alpha,\alpha\in F)\subset\sigma,\cup(\sigma_\alpha,\alpha\in F)\neq\sigma)$

when

$\cup\,(s_{\lambda_\alpha}\ \alpha \in F) \subset s_\lambda,$	$\cup\,(S_{\sigma_\alpha},\ \alpha \in F) \subset S_\sigma,$
$(\cup\,(s_{\lambda_\alpha},\ \alpha \in F) \subset s_{\lambda_\alpha},\ \cup\,(s_{\lambda_\alpha},\ \alpha \in F) \neq s_\lambda)$	$(\cup\,(S_{\sigma_\alpha},\ \alpha \in F) \subset S_\alpha, \cup\,(S_{\sigma_\alpha},\ \alpha \in F) \neq S_\sigma$

Def. 5. We will say that an

addition algorithm σ	convergence algorithm λ

is *equivalent* to

a convergence algorithm, λ	an addition algorithm, σ

and we will write sometimes

$\sigma = \lambda$	$\lambda = \sigma$

when the two following conditions are verified

$\sum_{n=0}^{\infty} a_n \in S_\sigma \rightleftarrows \{s_n\} \in s_\lambda$ (1)	$\{s_n\} \in s_\lambda \rightleftarrows \sum_{n=0}^{\infty} a_n \in S_\sigma$ (1')
$\sigma \sum_{n=0}^{\infty} a_n = \lambda \lim s_n$	$\lambda \lim s_n = \sigma \sum_{n=0}^{\infty} a_n$ (2')

being

$s_n = \sum_{\nu=0}^{n} a_\nu$ for $n = 0, 1, 2, \ldots$ (3)	$a_0 = s_0$ and $a_n = \Delta s_{n-1}$ for $n = 1, 2, \ldots$ (3')

Prop. 2. (Converse property). *If :*

$\sigma = \lambda$ | $\lambda = \sigma$

it is

$\lambda = \sigma$ | $\sigma = \lambda$

Proof. If

(1) and (2) | (1′) and (2′)

are verified for the

$\{s_n\}$ | $\{a_n\}$

defined in

(3), | (3′),

are verified

(1′) and (2′) | (1) and (2)

for the

$\sum_{n=0}^{\infty} a_n$ | $\{s_n\}$

defined in

(3′) | (3)

Prop. 3. (Transitive property). *If:*

$\sigma = \lambda,\ \sigma' = \lambda'$ and $\lambda = \lambda'$ | $\lambda = \sigma,\ \lambda' = \sigma'$ and $\sigma = \sigma'$

also

$\sigma = \sigma'$ | $\lambda = \lambda'$

Proof. Being, by assumption

$\mathbf{s}_\lambda = \mathbf{s}_{\lambda'}$ and $\lambda \lim s_n = \lambda' \lim s_n$ for $\{s_n\} \in \mathbf{s}_\lambda$ | $\mathbf{S}_\sigma = \mathbf{S}_{\sigma'}$ and $\sigma \sum_{n=0}^{\infty} a_n = \sigma' \sum_{n=0}^{\infty} a_n$ for $\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma$

it results

$\mathbf{S}_\sigma = \mathbf{S}_{\sigma'}$ and $\sigma \sum_{n=0}^{\infty} a_n = \sigma' \sum_{n=0}^{\infty} a_n$ for $\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma$ | $\mathbf{s}_\lambda = \mathbf{s}_{\lambda'}$ and $\lambda \lim s_n = \lambda' \lim s_n$ for $\{s_n\} \in \mathbf{s}_\lambda$

by virtue of

(1) and (2) | (1′) and (2′)

Prop. 4. *If:*

$\lambda \subset \lambda'$ (*or* $\lambda \subset \lambda'$ *but* $\lambda \neq \lambda'$) | $\sigma \subset \sigma'$ (*or* $\sigma \subset \sigma'$ *but* $\sigma \neq \sigma'$)

***and**, on the other hand*

$\sigma = \lambda$ *and* $\sigma' = \lambda'$, | $\lambda = \sigma$ *and* $\lambda' = \sigma'$

we have

$\sigma \subset \sigma'$ (*or* $\sigma \subset \sigma'$ *but* $\sigma \neq \sigma'$) | $\lambda \subset \lambda'$ (*or* $\lambda \subset \lambda'$ *but* $\lambda \neq \lambda'$)

Proof. Being

$s_\lambda \subset s_\lambda'$ (*or* $s_\lambda \subset s_\lambda'$ but $s_\lambda \neq s_\lambda'$),	$S_\sigma \subset S_\sigma'$ (or $S_\sigma \subset S_\sigma'$ but $S_\sigma \neq S_\sigma'$),

it results

$S_\sigma \subset S_\sigma'$ (or $S_\sigma \subset S_\sigma'$ but $S_\sigma \neq S_\sigma'$)	$s_\lambda \subset s_\lambda'$ (or $s_\lambda \subset s_\lambda'$ but $s_\lambda \neq s_\lambda'$)

after

(1) and (2)	(1′) and (2′)

Formals laws

Additive law

a) If	a′) If
$\{s_n\} \in s_\lambda$ and $\{t_n\} \in s_\lambda$,	$\sum_{n=0}^{\infty} a_n \in S_\sigma$ and $\sum_{n=0}^{\infty} b_n \in S_\sigma$
also	
$\{s_n + t_n\} \in s_\lambda$	$\sum_{n=0}^{\infty} (a_n + h_n) \in S_\sigma$
and we have	
λ lim. $(s_n + t_n) = \lambda \lim s_n + \lambda \lim t_n$	$\sigma \sum_{n=0}^{\infty} (a_n + b_n) = \sigma \sum_{n=0}^{\infty} a_n + \sigma \sum_{n=0}^{\infty} b_n$

Homogeneity

b) If	b′) If
$\{s_n\} \in s_\lambda$,	$\sum_{n=0}^{\infty} a_n \in S_\sigma$,
also	
$\{c\, s_n\} \in s_\lambda$	$\sum_{n=0}^{\infty} c\, a_n \in S_\sigma$
and it is	
$\lim (c\, s_n) = c.\ \lambda \lim.\ s_n$	$\sigma \sum_{n=0}^{\infty} (c\ a_n) = c.\ \sigma \sum_{n=0}^{\infty} a_n$

whichever the constant c may be.

Initial term

c) If	c′) If
$\{s_n\} \in s_\lambda$ and $\{s_{n+1}\} \in s_\lambda$,	$\sum_{n=0}^{\infty} a_n \in S_\sigma$ and $\sum_{n=1}^{\infty} a_n \in S_\sigma$,
being	
$s_0 = o$,	$a_0 = o$

we have

$$\lambda \lim s_n = \lambda \lim s_{n+1}$$

$$\sigma \sum_{n=0}^{\infty} a_n = \sigma \sum_{n=1}^{\infty} a_n$$

Product law

d) If

$$\{s_n\} \in \mathbf{s}_\lambda,\ \{t_n\} \in \mathbf{s}_\lambda \text{ and } \{z_n\} \in \mathbf{s}_\lambda$$

being

$$z_n = s_n t_n \text{ for } n = 0, 1, 2, \ldots,$$

we have

$$\lambda \lim z_n = \lambda \lim s_n . \lambda \lim t_n$$

d') If

$$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma,\ \sum_{n=0}^{\infty} b_n \in \mathbf{S}_\sigma \text{ and } \sum_{n=0}^{\infty} c_n \in \mathbf{S}_\sigma$$

$$c_n = a_n * b_n = \sum_{\nu=0}^{n} a_\nu b_{n-\nu} \text{ for } n = 0, 1, 2, \ldots,$$

$$\sigma \sum_{n=0}^{\infty} c_n = \sigma \sum_{n=0}^{\infty} a_n . \sigma \sum_{n=0}^{\infty} b_n$$

d'$_1$) If

$$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma,\ \sum_{n=0}^{\infty} b_n \in \mathbf{S}_\sigma \text{ and } \sum_{n=0}^{\infty} c'_n \in \mathbf{S}_\sigma,$$

being

$$c'_0 = o \text{ and } c'_{n+1} = a_n * b_n \text{ for } n = 0, 1, 2, \ldots,$$

we have

$$\sigma \sum_{n=0}^{\infty} c'_n = \sigma \sum_{n=0}^{\infty} a_n . \sigma \sum_{n=0}^{\infty} b_n$$

Abel law

e$_1$) If

$$\{s_n\} \in \mathbf{s}_\lambda, \qquad (4)$$

also

$$\{s_n x^n\} \in \mathbf{s}_\lambda \qquad (5)$$

$$\left\{ \sum_{\nu=0}^{n} s_\nu x^\nu \right\} \in \mathbf{s}_\lambda \qquad (6)$$

and we have

$$\lambda \lim (s_n x^n) = 0 \qquad (7)$$

e'$_1$) If

$$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma, \qquad (4')$$

$$\sum_{n=0}^{\infty} a_n x^n \in \mathbf{S}_\sigma \qquad (5')$$

$$\sum_{n=0}^{\infty} (1 * a_n)\, x^n \in \mathbf{S}_\sigma \qquad (6')$$

$$\sigma \sum_{n=0}^{\infty} a_n x^n = (x-1) . \sigma \sum_{n=0}^{\infty} (1 * a_n)\, x^n \qquad (7')$$

for any $x \in]0, 1[$

e_2) If

$$\{s_n\} \in s_\lambda$$

and $x \in]0, 1[$, we have

$$\lim_{x\to 1-} [(1-x).\, \lambda \lim \sum_{\nu=0}^{n} s_\nu x^\nu] = \lambda \lim.\, s_n \quad (8)$$

e'_2) If

$$\sum_{n=0}^{\infty} a_n \in S_\sigma$$

$$\lim_{x\to 1-} \sigma \sum_{n=0}^{\infty} a_n x^n = \sigma \sum_{n=0}^{\infty} a_n \quad (8')$$

f) The limit

$$\lambda \lim s_n$$

f′) The sum

$$\sigma \sum_{n=0}^{\infty} a_n$$

is a linear continuous functional in the space

s_λ | S_σ

with a topology.

Remark.—When c′is verified, d′ and d'_1 are evidently equivalent.

Through the laws

a and b, the law c | a' and b', the law c'

can be generalized for any initial term as follows:

c_1) If

$$\{s_n\} \in s_\lambda \text{ and } \{s_{n+1}\} \in s_\lambda,$$

we have

$$\lambda \lim s_n = \lambda \lim s_{n+1}$$

whichever the initial term may be.

c'_1) If

$$\sum_{n=0}^{\infty} a_n \in S_\sigma \text{ and } \sum_{n=1}^{\infty} a_n \in S_\sigma,$$

$$\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n$$

Proof. Writing

$s'_0 = 0$ and $s'_n = s_n$ for $n = 1, 2, \ldots$ | $a'_0 = 0$ and $a'_n = a_n$ for $n = 1, 2, \ldots$

$s_n'' = s_0$ and $s_n'' = 0$ for $n = 1, 2, \ldots$ | $a''_0 = a_0$ and $a''_n = 0$ for $n = 1, 2, \ldots$

we have evidently

$s_n = s'_n + s''_n$ for $n = 0, 1, 2, \ldots;$ | $a_n = a'_n + a''_n$ for $n = 0, 1, 2, \ldots$

but us

$$\{s_n''\} \in s_{\lambda_0} \subset s_\lambda,\ \lambda \lim s''_n = \lim_{n\to\infty} s''_n = 0$$

$$\sum_{n=0}^{\infty} a_n'' \in S_{\sigma_0} \subset S_\sigma,\quad \lambda \sum_{n=0}^{\infty} s_n'' = \sum_{n=0}^{\infty} s_n'' = a_0$$

it follows from

a and b, | a' and b',

that also

$\{s'_n\} \epsilon \, \mathbf{s}_\lambda;$ | $\sum_{n=0}^{\infty} a'_n \epsilon \, \mathbf{S}_\sigma;$

wherefrom, being nul its initial term, it follows according to

c: $\lambda \lim s_n = \lambda \lim s'_n + \lambda \lim s''_n = \lambda \lim s_{n+1} + \lambda_0 \lim. s''_n = \lambda \lim. s_{n+1}$ | c': $\sigma \sum_{n=0}^{\infty} a_n = \sigma \sum_{n=0}^{\infty} a'_n + \sigma \sum_{n=0}^{\infty} a''_n = \sigma \sum_{n=1}^{\infty} a'_n + \sigma_0 \sum_{n=0}^{\infty} a''_n = \sigma \sum_{n=1}^{\infty} a_n + a_0$

From the laws

e_1 and e_2, | e'_1 and e'_2,

can be deduced the following one, which may replace both in the applications.

e) If $\{s_n\} \epsilon \, \mathbf{s}_\lambda$ | e') If $\sum_{n=0}^{\infty} a_n \epsilon \, \mathbf{S}_\sigma$

it is also

$\left\{ \sum_{\nu=0}^{n} s_\nu x^\nu \right\} \epsilon \, \mathbf{s}_\lambda$ | $\sum_{n=0}^{\infty} a_n x^n \epsilon \, \mathbf{S}_\sigma$

for every $x \epsilon]\, 0, 1[$, and we have

$\lim_{x \to 1-} [(1-x). \lambda \lim \sum_{\nu=0}^{n} s_\nu x^\nu] = \lambda \lim s_n$ | $\lim_{x \to 1-} \sigma \sum_{n=0}^{\infty} a_n x^n = \sigma \sum_{n=0}^{\infty} a_n$

Proof. The assumption is the same as in

e_1 and e_2 | e'_1 and e'_2

and the conclusions are

(6) of e_1 and (8) of e_2 | (6') of e'_1 and (8') of e'_2

Whichever may be the formal laws taken in Def. 1 of the algorithms, the laws

a, b, c, e_1, e_2 and f | a', b', c', e'_1, e'_2 and f'

are surely transformed in the

a', b', c', e'_1, e'_2 and f' | a, b, c, e_1, e_2 and f

in the equivalence (Def. 5) of

a convergence algorithm | an addition algorithm

with another

addition one, | convergence one,

as expressed in the following proposition

Prop. 5. *If*

a convergence algorithm λ | *an addition algorithm* σ

verifies the formal laws

a, b, c, e_1, e_2 and f | a', b', c', e'_1, e'_2 and f'

with a certain topology

T_λ, | T_σ

every

addition algorithm σ | *convergence algorithm* λ

equivalent (Def. 5) *to the*

λ, $\sigma = \lambda$, | σ, $\lambda = \sigma$,

verifies the homolous laws

a', b', c', e'_1, e'_2 and f' | a, b, c, e_1, e_2 and f

with the topology

T_σ, | T_λ

obtained by taking as neighborhood of a

series $\sum_{n=0}^{\infty} a_n^{(0)} \in \mathbf{S}_\sigma$ *in* $\mathbf{S}_\sigma$, | *sequence* $\{s_n^{(0)}\} \in \mathbf{s}_\lambda$ *in* $\mathbf{s}_\lambda$,

the set of the

series $\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma$ | *sequences* $\{s_n\} \in \mathbf{s}_\lambda$

whose

sequences $\{s_n\}$ | *series* $\sum_{n=0}^{\infty} a_n$

given by

(3) | (3')

belongs to the neighborhood in

$\mathbf{s}_\lambda$, | $\mathbf{S}_\sigma$,

after the topology

T_λ | T_σ,

with which is verified

f, | f',

of the

sequence $\{s_n^{(0)}\} \in \mathbf{s}_\lambda$ | *series* $\sum_{n=0}^{\infty} a_n^{(0)} \in \mathbf{S}_\sigma$

obtained by applying

(3) | (3')

for the sequence

$a_n = a_n^{(0)}$	$s_n = s_n^{(0)}$

for $n = 0, 1, 2, \ldots$

Proof. Being

$s_0 \subset s_\lambda$ and $s_0 \neq s_\lambda$,	$S_0 \subset S_0$ and $S_0 \neq S_0$,

it results from

(1) : $S_0 \subset S_\sigma$ and $S_0 \neq S_\sigma$	(1′) : $s_\sigma \subset s_\lambda$ and $s_0 \neq s_\lambda$

Furthermore, if

$\sum_{n=0}^{\infty} a_n \in S_\sigma$,	$\{s_n\} \in s_\lambda$,

i. e., after

(1), $\{s_n\} \in s_\lambda$	(1′), $\sum_{n=0}^{\infty} a_n \in S_\sigma$

we deduce from

(2),	(2′)

being

λ	σ

an extension of

λ_0 :	σ_0 :
$\sigma \sum_{n=0}^{\infty} a_n = \lambda \lim s_n = \lambda_0 \lim s_n = \lim_{n\to\infty} s_n = \sum_{n-0}^{\infty} a_n = \sigma_0 \sum_{n=0}^{\infty} a_n$	$\lambda \lim s_n = \sigma \sum_{n=0}^{\infty} a_n = \sigma_0 \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n = \lambda_0 \lim s_n$

that is

σ	λ

is an extensión of

σ_0	λ_0

Now, we are going to proof the formal laws

a′) Let be	a) Let be
$\sum_{n=0}^{\infty} a_n \in S_\sigma$ and $\sum_{n=0}^{\infty} b_n \in S_\sigma$,	$\{s_n\} \in s_\lambda$ and $\{t_n\} \in s_\lambda$,

i. e., after

(1), $\{s_n\} \in s_\lambda$ and $\{t_n\} \in s_\lambda$	(1′), $\sum_{n=0}^{\infty} a_n \in S_\sigma$ and $\sum_{n=0}^{\infty} b_n \in S_\sigma$

being

$$s_n = \sum_{\nu=0}^{n} a_\nu \text{ and } t_n = \sum_{\nu=0}^{n} b_\nu \text{ for } n=0, 1, 2, \ldots$$

$$a_0 = s_0,\ a_n = \Delta s_{n-1} \text{ and } b_0 = t_0,\ b_n = \Delta t_{n-1} \text{ for } n = 1, 2, \ldots$$

Since

$$\sum_{\nu=0}^{n} (a_\nu + b_\nu) = s_n + t_n \text{ for } n = 0, 1, 2, \ldots$$

we have, after

(2) and a :

$$\sigma \sum_{n=0}^{\infty} (a_n + b_n) = \lambda \lim (s_n + t_n) = \lambda \lim s_n + \lambda \lim t_n = \sigma \sum_{n=0}^{\infty} a_n + \sigma \sum_{n=0}^{\infty} b_n$$

(2′) and a′ :

$$\lim (s_n + t_n) = \sigma \sum_{n=0}^{\infty} (a_n + b_n) = \sigma \sum_{n=0}^{\infty} a_n + \sigma \sum_{n=0}^{\infty} b_n = \lambda \lim s_n + \lambda \lim t_n$$

which is the

a′) | a)

b′) If

$$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma,$$

b) If

$$\{s_n\} \in \mathbf{s}_\lambda,$$

i. e., after

$$(1),\ \{s_n\} \in \mathbf{s}_\lambda \qquad (1'),\ \sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma$$

also, after

$$\text{b},\ b\{c\ s_n\} \in \mathbf{s}_\lambda \qquad \text{b}',\ \sum_{n=0}^{\infty} c\ a_n \in \mathbf{S}_\sigma$$

whichever the constant c may be, that is, after

$$(1),\ \sum_{n=0}^{\infty} c\ a_n \in \mathbf{S}_\sigma \qquad (1'),\ \{c\ s_n\} \in \mathbf{s}_\lambda$$

and besides, we have, after

b :

$$\sigma \sum_{n=0}^{\infty} (c\ a_n) = \lambda \lim (c\ s_n) = c.\ \lambda \lim s_n = c.\ \sigma \sum_{n=0}^{\infty} a_n$$

b′ :

$$\lambda \lim (c\ s_n) = \sigma \sum_{n=0}^{\infty} c\ a_n = c.\ \sigma \sum_{n=0}^{\infty} a_n = c.\ \lambda \lim s_n$$

which is just

b′

c′) Let be

$$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma \text{ and } \sum_{n=1}^{\infty} a_n \in \mathbf{S}_\sigma \qquad (9)$$

being

$$a_0 = o$$

By according with

$$(3):\ s_n = \sum_{\nu=0}^{n} a_\nu \text{ for } n = 0,\ 1, \ldots$$

$$s'_n = \sum_{\nu=1}^{n+1} a_\nu \text{ for } n = 0,\ 1, \ldots$$

we have, after

$$(1) \text{ and } (9),\ \{s_n\} \in \mathbf{s}_\lambda \text{ and } \{s'_n\} \in \mathbf{s}_\lambda$$

But it is abviously

$$s'_n = s_{n+1}$$

b

c) Let be

$$\{s_n\} \in \mathbf{s}_\lambda \text{ and } \{s_{n+1}\} \in \mathbf{s}_\lambda \qquad (9')$$

$$s_0 = o$$

$$(3'):\ a_0 = o,\ a_n = \Delta\, s_{n-1} \text{ for } n = 1,\ 2, \ldots$$

$$a'_0 = s_1,\ a_n = \Delta\, s_n \text{ for } n = 1,\ 2, \ldots$$

$$(1') \text{ and } (9'),\ \sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma \text{ and } \sum_{n=0}^{\infty} a'_n \in \mathbf{S}_\sigma$$

$$a'_n = a_{n+1}$$

for $n = 0,\ 1,\ 2, \ldots$; whence it results by successive applying of:

(2), c and (2):

$$\sigma \sum_{n=0}^{\infty} a_n = \lambda \lim s_n = \lambda \lim s_{n+1} = \lambda \lim s'_n = = \sigma \sum_{n=1}^{\infty} a_n$$

(2′), c' and (2′):

$$\lambda \lim s_n = \sigma \sum_{n=0}^{\infty} a_n = \sigma \sum_{n=1}^{\infty} a_n = \sigma \sum_{n=0}^{\infty} a'_n = = \lambda \lim s_{n+1}$$

e′₁) Let be

$$\{s_n\} \in \mathbf{s}_\lambda$$

e₁) Let be

$$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma$$

The conclusion

(6) of e_1, | (6′) of e'_1,

is exactly the

(6′) of e′₁ | (6) of e_1

after

(1′) and (3′) | (1) and (3)

From Abel's identity, or partial addition:

$$\sum_{\nu=0}^{n+1} a_\nu x^\nu = (1 - x) \sum_{\nu=0}^{n} s_\nu\, x^\nu + s_{n+1}\, x^{n+1},$$

$$s_n\, x^n = \sum_{\nu=0}^{n} a_\nu\, x^\nu - (1 - x) \sum_{\nu=0}^{n-1} (1 * a_\nu)\, x^\nu,$$

and through a new application of

(6), b, (5), b, a and (1),	(6′), (1′), b, (5′), (1′), a and b,

it follows

$\sum_{n=0}^{\infty} a_n x^n \in \mathbf{S}_\sigma$	$\{s_n x^n\} \in \mathbf{s}_\lambda$

which is

(5′) of e'_1;	(5) of e_1;

and besides, by virtue of

(2) and (7) of e_1,	(2′) and (7′) of e'_1

we have

$$\sigma \sum_{n=0}^{\infty} a_n x^n = (1-x)\ \lambda \lim \sum_{\nu=0}^{n} s_\nu x^\nu + \lambda \lim (s_{n+1} x^{n-1}) = (1-x).\ \lambda \lim \sum_{\nu=0}^{n} s_\nu x^\nu = (1-x).\ \sigma \sum_{n=0}^{\infty} s_n x^n = (1-x).\ \sigma \sum_{n=0}^{\infty} (1 * a_n)\, x^n \tag{10}$$

$$\lambda \lim (s_n x^n) = \lambda \lim \sum_{\nu=0}^{n} a_\nu x^\nu - (1-x).\ \lambda \lim \sum_{\nu=0}^{n-1} (1 * a_\nu)\, x^\nu = \sigma \sum_{n=0}^{\infty} a_n x^n - (1-x).\ \sigma \sum_{n=0}^{\infty} (1 * a_n)\, x^n = 0 \tag{10'}$$

which is just

(7′) of e'_1	(7) of e_1.

e′$_1$) From

(10), (8) of e_2 and (2),	(10′), (2′), (8′) of e'_2 and (2′)

it results

$$\lim_{x\to 1^-} \sigma \sum_{n=0}^{\infty} a_n x^n = \lim_{x\to 1^-} [(1-x).\ \lambda \lim \sum_{\nu=0}^{n} s_\nu x^\nu] = \lambda \lim s_n = \sigma \sum_{n=0}^{\infty} a_n$$

$$\lim_{x\to 1^-} [(1-x).\ \lambda \lim \sum_{\nu=0}^{n} s_\nu x^\nu] = \lim_{v\to 1^-} [(1-x).\ \lambda \lim \sum_{\nu=0}^{n} (1 * a_n)\, x^n] = \lim_{x\to 1^-} [\lambda \lim \sum_{\nu=0}^{n} a_\nu x^\nu] = \lim_{x\to 1^-} \sigma \sum_{n=0}^{\infty} a_n x^n = \sigma \sum_{n=0}^{\infty} a_n = \lambda \lim s_n$$

which is the conclusion

(8′) of e'_2	(8) of e_2

f′) After

f,	f,

to every ϵ-neighborhood of the number

$\sigma \sum_{n=0}^{\infty} a_n^{(0)} = \lambda \lim s_n^{(0)}$	$\lambda \lim s_n^{(0)} = \sigma \sum_{n=0}^{\infty} a_n^{(0)}$

it corresponds a δ-neighborhood of	
$\{s_n^{(0)}\}$	$\sum_{n=0}^{\infty} a_n^{(0)}$
in the topology	
T_λ	T_σ
such that if	
$\{s_n\} \epsilon\ \delta, \qquad \lambda \lim s_n \epsilon\ \varepsilon;$ (11)	$\sum_{n=0}^{\infty} a_n \epsilon\ \delta, \qquad \sigma \sum_{n=0}^{\infty} a_n \epsilon\ \varepsilon;$ (11′)
and to this one, another δ' of	
$\sum_{n=0}^{\infty} a_n^{(0)}$	$\{s_n^{(0)}\}$
in the topology	
T_σ	T_λ
such that if	
$\sum_{n=0}^{\infty} a_n \epsilon\ \delta',$	$\{s_n\} \epsilon\ \delta'$
we have by definition	
$\{s_n\} \epsilon\ \delta;$	$\sum_{n=0}^{\infty} a_n \epsilon\ \delta;$
therefore, after	
(11) and (2)	(11′) and (2′)
we have	
$\sigma \sum_{n=0}^{\infty} a_n = \lambda \lim s_n \epsilon\ \varepsilon$	$\lambda \lim s_n = \sigma \sum_{n=0}^{\infty} a_n \epsilon\ \varepsilon$
That is to say, to every ε-neighborhood of	
$\sigma \sum_{n=0}^{\infty} a_n^{(0)}$	$\lambda \lim s_n^{(0)}$
it corresponds another δ' of	
$\sum_{n=0}^{\infty} a_n^{(0)}$	$\{s_n^{(0)}\}$
in the topology	
T_σ	T_λ
such that if	
$\sum_{n=0}^{\infty} a_n \epsilon\ \delta'$	$\{s_n\} \epsilon\ \delta'$

we have

$\sigma \sum_{n=0}^{\infty} a_n \in \mathbf{\varepsilon}$;	$\lambda \lim s_n \in \mathbf{s}$;

which means that the functional

$\sigma \sum_{n=0}^{\infty} a_n$	$\lambda \lim s_n$

is continuous in

$\mathbf{S}_\sigma$	$\mathbf{s}_\lambda$

with the topology

T_σ	T_λ

transformed from the topology

T_λ	T_σ

Remark. 1. In this Prop. 5 we have to do without the product laws, which are not deducible from an algorithm to the other one, since obviously

$$\sum_{\nu=0}^{n} a_\nu . \sum_{\nu=0}^{n} b_\nu \neq \sum_{\nu=0}^{n} (a_\nu * b_\nu),$$

and this is just the cause of troubles involved in the product law for summation algorithms, including those for convergent series.

2. In stead of these product laws and as auxiliaires for the converse deduction of Abel's laws

e or e_1	e' or e'_1

we have introduced the converse laws e_1 and e'_1, whose last conclusions

(7)	(7')

are two particular cases of the said product laws, since we have

$\lambda \lim (s_n x^n) = \lambda \lim s_n . \lambda \lim x^n =$ $= \lambda \lim s_n . \lambda_0 \lim x^n = 0$	$\sigma \sum_{n=0}^{\infty} (1 * a_n) x^n = \sigma \sum_{n=0}^{\infty} (x^n * a_n x^n) =$ $= \sigma \sum_{n=0}^{\infty} x^n . \sigma \sum_{n=0}^{\infty} a_n x^n = \sigma_0 \sum_{n=0}^{\infty} x^n . \sigma \sum_{n=0}^{\infty} a_n x^n =$ $= \frac{1}{1-x} . \sigma \sum_{n=0}^{\infty} a_n x^n$

2. *Linear algorithms*

It is convenient to fix some terminology through the following definitions:

Def. 1. We will call ***linear algorithms*** of

convergence, λ	addition, μ

those defined by means of a matrix, or double sequence

$\{\lambda_{r,t}\}$	$\{\mu_{r,t}\}$

with two indexes $r = 0, 1, 2, \dots$ and $t = 0, 1, 2, \dots$ as in

[1, 90]; | [7, 42];

and *linear algorithms* of

convergence, λ | addition, μ

with continuous parameter, t, or more shortly *continuous* algorithms, those defined by means of a *factors* sequence of

$\{\lambda_r(t)\}$ - *convergence* | $\{\mu_r(t)\}$ - *addition*

with an $r = 0, 1, 2, \dots$ and variable t within a set E with the adherence point t_0, as in [12, 193-197], where it was analized in detail the relation between them.

Permanent linear algorithms are evidently distributive algorithms (Def. 1, no. 1).

Def. 2. We call *transforms* of a

sequence $\{a_n\}$ (1) | series $\sum_{n=0}^{\infty} a_n$ (1′)

by means of a matrix

$\{\lambda_{t,n}\}$ | $\{\mu_{t,n}\}$

or by means of a sequence of functions

$\{\lambda_n(t)\}$ | $\{\mu_n(t)\}$

to the sequence

$\left\{\sum_{n=0}^{\infty} \lambda_{t,n}\, a_n\right\}$ (2) | $\left\{\sum_{n=0}^{\infty} \mu_n(t)\, a_n\right\}$ (2′)

or the function of t, defined by the series

$\sum_{n=0}^{\infty} \lambda_n(t)\, \alpha_n$ (3) | $\sum_{n=0}^{\infty} \mu_n(t)\, a_n$ (3′)

respectively.

Prop. 1. *If the*

sequences (1) | *series* (1′)

form a vectorial space, their transforms

(2) or (3) | (2′) or (3′)

do it also.

Proof. Because the transformation is linear.

Def. 3. We will say that a linear algorithm of

convergence, λ | addition, μ

verifies the *principle of identity*, when the only

sequence $\{a_n\} \in s_\lambda$ | series $\sum_{n=0}^{\infty} a_n \in S\mu$

having all its transforms null

(2)	(2′)

for $t = 0, 1, 2, \ldots$ or the

(3)	(3′)

for $t \varepsilon E$, is the identically zero one

$$a_n = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

and when furthermore, in the second case, the set E is compact and the transforms

(2)	(2′)

of every

sequence $\{s_n\} \epsilon \mathbf{s}_\lambda$	series $\sum_{n=0}^{\infty} a_n \epsilon \mathbf{S}_\mu$

are continuous functions in $E\text{-}t_0 = E \cap \complement\, t_0$.

Prop. 2. *If a linear algorithm verifies the identity principles, the set*

$\mathbf{s}_\lambda$	$\mathbf{S}_\mu$

and that of the transforms correspond one-to-one each other; the vectorial space of the transforms

(2) *or* (3)	(2′) *or* (3′)

is a vectorial subspace $\mathbf{c}_1$ *or* $\mathbf{C}_1$, *respectively for the space* $\mathbf{c}$ *of the convergent sequences or for the space* $\mathbf{C}$ *of the continuous functions within the compact set* E*; and we may define a metric in the set*

$\mathbf{s}_\lambda$	$\mathbf{S}_\mu$

in such a way that the latter resulte isometric with the subspace $\mathbf{c}_1 \subset \mathbf{c}$ *or with the* $\mathbf{C}_1 \subset \mathbf{C}$, *adopting as norm for every*

sequence $\{s_n\} \epsilon \mathbf{s}_\lambda$,	*series* $\sum_{n=0}^{\infty} a_n \epsilon \mathbf{S}_\sigma$

that of its transform in $\mathbf{c}_1$ *or* $\mathbf{C}_1$.

Proof. The correspondence between.

$\mathbf{s}_\lambda$ and $\mathbf{c}_1$ or $\mathbf{C}_1$	$\mathbf{S}_\mu$ and $\mathbf{c}_1$ or $\mathbf{C}_1$

is obviously univocal; and one-to-one becuse of the identity principle.

We have $\mathbf{c}_1 \subset \mathbf{c}$, because there exists

$\lambda \lim s_n = \lim\limits_{t\to\infty} \sum_{n=0}^{\infty} \lambda_{t,n}\, s_n < \infty \quad (5)$	$\sigma \sum_{n=0}^{\infty} a_n = \lim\limits_{t\to\infty} \sum_{n=0}^{\infty} \mu_{t,n}\, a_n < \infty \quad (5')$

and $C_1 \subset C$ because existing

$\lambda \lim s_n = \lim\limits_{t\to t_0} \sum_{n=0}^{\infty} \lambda_n(t)\, s_n < \infty,\ t \epsilon E, \ (6)$	$\sigma \sum_{n=0}^{\infty} a_n = \lim\limits_{t\to t_0} \sum_{n=0}^{\infty} \mu_n(t)\, a_n,\ t \epsilon E \quad (6')$

the transform is furthermore continuous in $E\text{-}t_0$; therefore this transform is a cotinous function in the whole set E, adopting this limit as value at t_0.

Finally, the norm has identical property by virtue of the identity principle; and the spaces result obviously isometric, since they are vectorial spaces with equal norms at homologous points.

Def. 4. The metric defined in Prop. 2 and the topology determined in

$\mathbf{s}_\lambda$ | $\mathbf{S}_\mu$

will be called *specific* for the algorithm (*).

Prop. 3. *If an algorithm verifies the identity principle (Def. 3) it satisfies the continuity law*

f | f'

with its specific topology.

Proof. The limit

(5) or (6) | (5′) or (6′)

is, of course, a continuous functional in $\mathbf{c}_1$ or $\mathbf{C}_1$ with the metric of $\mathbf{c}$ or $\mathbf{C}$, respectively, and, therefore, in

$\mathbf{s}_\lambda$ | $\mathbf{S}_\mu$

with the specific metric of the algorithm, because then, both spaces are isometric; and therefore it shall be continuous with the specific topology for the algorithm (*).

The definitions of reversible and perfect convergence algorithms [1, 90] could be generalized for continuous algorithms. But the extension of fundamental theorems requires a detailed study, specially the [1, Lemma 4, 93], based on the [1, Theor. 10, 47] which we are trying to avoid here, through the algorithms with matrix subordinated to a continuous algorithm.

Def. 5. Being given a continuous linear algorithm of

convergence, λ | addition, μ

with the factors

$\{\lambda_n(t)\}$, | $\{\mu_n(t)\}$,

we call *subordinate algorithm through a sequence* $\{r_t\} \in E$ with limit t_0 and it will be designed sometimes by

$\lambda(r_t)$, | $\mu(r_t)$,

to the algorithm of

convergence | addition

with the matrix

$\{\lambda_n(r_t)\}$ | $\{\mu_n(r_t)\}$

for $t = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$

Prop. 4. *Let*

λ | μ

(*) Somewhere else we will analize the relation between this metric and the topology defined by Erdös and Piranian [6, 139-148] by means of the Toeplitz matrices in the set of bounded sequences.

be a continuous algorithm of

convergence	addition

In order that there exist

$\lambda \lim s_n,$ (7)	$\mu \sum_{n=0}^{\infty} a_n,$ (7′)

that is,

$\{s_n\} \in \mathbf{s}_\lambda,$	$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\mu,$

it is necessary and sufficient that there exist

$\lambda(r_t) \lim s_n,$ (8)	$\mu(r_t) \sum_{n=0}^{\infty} a_n,$ (8′)

that is,

$\{s_n\} \in \mathbf{s}_{\lambda(r_t)},$	$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_{\mu(r_t)}$

for every sequence $\{r_t\} \in E$ *with limit* t_0. *Evidently :*

all these limits (8)	*all these sums* (8′)

result equal each other and

to the limit (7)	*to the limit* (7′)

Using symbols

$\mathbf{s}_\lambda = \cap (\mathbf{s}_{\lambda(r_t)}, \{r_t\} \in E, \lim_{t \to \infty} r_t = t_0)$ (9)	$\mathbf{S}_\mu = \cap (\mathbf{S}_{\mu(r_t)}, \{r_t\} \in E, \lim_{t \to \infty} r_t = t_0)$ (9′)
$\lambda \lim s_n = \lambda(r_t) \lim s_n$ (10)	$\mu \sum_{n=0}^{\infty} a_n = \mu(r_t) \sum_{n=0}^{\infty} a_n$ (10′)

for every sequence $\{r_t\} \in E$.

In particular, if the compact set E, *with the adherence point* t_0 *is the real positive semi-axis* $[0, +\infty]$ *and* $t = +\infty$, *the preceding conclusions remain valid, but limited to the increasing sequences* $\{r_t\}$ *with limit* $+\infty$ *and all their terms larger than any prefixed constant.*

Proof. They are abvious consequences of the elemental properties of arithmetical and functional limits.

Cors. 1. *Any continuous algorithm is weaker than any of oits subordinate ones, i. e.*

$\lambda \subset \lambda(r_t)$	$\mu \subset \mu(r_t)$

for any sequence $\{r_t\} \in E$ *with limit* t_0.

Proof. An obvious consequence from

(9) and (10)	(9′) and (10′)

2. If a continuous algorithm is permanent, all its subordinates are permanent too.

Proof. Immediate consequence from permanence definition :

$$\lambda_0 \subset \lambda,\ \lambda_0 \lim s_n = \lambda \lim s_n,$$

$$\mu_0 \subset \mu\ (\mu_0 = \sigma_0),\ \mu_0 \sum_{n=0}^{\infty} a_n = \mu \sum_{n=0}^{\infty} a_n$$

(Prop. 1, no. 1), Cor. 1 and

(10) | (10′)

3. *Moments method.*

We are still giving immediate definitions and propositions, which will be used later.

Def. 1. We use to call [12, 199] [7, 81]

summatory $\{\nu_n\}$ | integral $\{\mu_n\}$

algorithm of moments

to a permanent method [13, 34] of summation, defining the sum of a series $\sum_{n=0}^{\infty} a_n$ as follows

$$\nu \sum_{n=0}^{\infty} a_n = \sum_{t=0}^{\infty} a_t A_t < \infty \quad (1)$$

$$\mu \sum_{n=0}^{\infty} a_n = \int_0^{\infty} a(t)\, A(t)\, dt < \infty \quad (1')$$

where

$\{a_t\}$ is the *generatrix* sequence with the moments

$$\nu_n = \sum_{t=0}^{\infty} a_t\, t^{(n} < \infty$$

for $n = 0, 1, 2, \ldots$ and A_t the *associated sequence* formed by the values of the polynomials

$$A_t = \sum_{n=0}^{t} a_n \frac{t^{(n}}{n} \text{ for } t = 0, 1, 2, \ldots \quad (2)$$

$a(t)$ is the continuous function in $[0, +\infty]$

$$\mu_n = \int_0^{\infty} a(t)\, t^n\, dt < \infty$$

and $A(t)$ the *associated function* obtained by analytical continuation along the positive real semi-axis of the function holomorph in 0 defined by the series

$$A(t) = \sum_{t=0}^{\infty} a_n \frac{t^n}{n!} \quad (2')$$

Prop. 1. *Conversely, any*

sequence $\{A_t\}$ *is the associated* ν | *holomorph function,* $A(t)$, *in* 0, *analytical continuable along the whole positive real semi-axis is the associated* μ

of a series $\sum_{n=0}^{\infty}$ *with coefficients*

$$a_0 = A_0 = \Delta^0 A_0 \text{ and } a_n = \frac{\Delta^n A_0}{n!} \nu_n \quad (3)$$

$$a_0 = A(0) = A^{(0)}(0) \text{ and } a_n = \frac{A^{(n)}(0)}{n!} \mu_n \quad (3')$$

for $n = 1, 2, \ldots$

Proof. Since we have evidently:

$$\sum_{n=0}^{t} a_n \frac{t^{(n}}{\nu_n} = \sum_{n=0}^{t} \frac{t^{(n}}{n!} \Delta^n A_0 = \sum_{n=0}^{t} \binom{t}{n} \Delta^n A_0 = A_t \text{ for } t = 0,\ 1,\ 2,\ \ldots$$

$$\sum_{n=0}^{\infty} a_n \frac{t^n}{\mu_n} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^{(n)}(0) = A(t)$$

in a neighborhood of the origin and by continuation along the whole positive real semi-axis, since, by assumption, $A(t)$ is holomorph in 0 and continuable along this semi-axis.

Prop. 2. *In the set*

$\mathbf{S}_\nu$ | $\mathbf{S}_\mu$

of summable series with an algorithm of

summatory moments $\{\nu_n\}$ | *integral moments* $\{\mu_n\}$

it can be defined a metric by adopting as norm for every series

$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\nu$ | $\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\mu$

that of the product of the associated by the generatrix in the vectorial subspace

$\mathbf{u}_0 \subset \mathbf{u}$ (4) | $\mathbf{U}_0 \subset \mathbf{U}$ (4′)

formed by the said products, so that we have a space

$\mathbf{S}_\nu$ | $\mathbf{S}_\mu$

isometric with

$\mathbf{u}_0$ | $\mathbf{U}_0$

Proof. By virtue of formulae

(2) and (3), | (2′) and (3′),

there exists a one-to-one correspondence, without exception, between the set

$\mathbf{S}_\nu$ | $\mathbf{S}_\mu$

of summable series

ν | μ

and the set

$\mathbf{u}_0$ | $\mathbf{U}_0$

of the products

$\{\alpha_t A_t\}$ | $\alpha(t) A(t)$

of the generatrix,

$\{\alpha_t\}$ | $\alpha(t)$

of the algorithm, by the associated in

$\{A_t\}$, | $A(t)$,

of every series

$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\nu$	$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\mu$;

and in this correspondence, the identically zero series :

$$\sum_{n=0}^{\infty} a_n \text{ with } a_n = 0 \quad \text{for} \quad n = 0,\ 1,\ 2, \ldots$$

and the identically zero associated

$A_t = 0$ for $t = 0, 1, 2, \ldots$	$A(t) = 0$ for $0 \leq t < +\infty$

become homologous.

This set

$\mathbf{u}_0$	$\mathbf{U}_0$

is evidently a vectorial space, as well as the

$\mathbf{S}_\nu$	$\mathbf{S}_\mu$

But being convergent the

series (1),	integrals (1′)

the space

$\mathbf{u}_0$	$\mathbf{U}_0$

is contained in the

$\mathbf{u}$	$\mathbf{U}$

defined in

(Def. 2, No. 1, § 1).	(Def. 2′, No. 1, § 1).

Then, we can adopt as norm for every series

$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\nu$	$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\mu$

that of the homologous product

$\{a_t A_t\} \in \mathbf{u}_0 \subset \mathbf{u}$	$[a(t) A(t)] \in \mathbf{U}_0 \subset \mathbf{U}$

in the space

$\mathbf{u}$	$\mathbf{U}$

This norm verifies the identical property, since, as we have said, the identically zero series and the identically zero product are homologous and so, both spaces

$\mathbf{S}_\nu$ and $\mathbf{u}_0$	$\mathbf{S}_\mu$ and $\mathbf{U}_0$

become isometrical, because they are vectorial spaces with equal norms at homologous points.

Def. 2. The metric defined in the Prop. 2 and the topology determined by this metric in

$\mathbf{S}_\nu$	$\mathbf{S}_\mu$

will be called *specific* for the algorithm

ν	μ

Prop. 3. *The continuity law f′ become true for the moments algorithm with the specific topology of the latter.*

Proof. The value of the

series (1)	integral (1′)

being a continuous functional in the vectorial subspace

$\mathbf{u}_0 \subset \mathbf{u}$	$\mathbf{U}_0 \subset \mathbf{U}$

with the metric of the space

$\mathbf{u}$	$\mathbf{U}$

shall be also, according to Prop. 2, a continuous functional in

$\mathbf{S}_\nu$	$\mathbf{S}_\mu$

with the metric and, therefore, with the specific topology of

ν	μ

Def. 3. An algorithm μ of integral moments being given (Def. 1) we will call *subordinate algorithm* to the μ through a sequence $\{r_t\}$ of positive numbers with $+\infty$ limit, and we will design it sometimes by $\mu\ (r_t)$, to that one defining the sum of a series $\sum_{t=0}^{\infty} a_n$, as follows:

$$\mu\,(r_t) \sum_{n=0}^{\infty} a_n = \lim_{t\to\infty} \int_0^{r_t} \alpha\,(\nu)\, A\,(\nu)\, d\nu$$

Prop. 4. *In order to exist the sum:*

$$\mu \sum_{n=0}^{\infty} a_n, \text{ that is, } \sum_{n=0}^{\infty} a_n \in \mathbf{S}_\mu,$$

it is necessary and sufficient that there exist the sums

$$\mu\,(r_t) \sum_{n=0}^{\infty} a_n, \text{ that is, } \sum_{n=0}^{\infty} a_n \in \mathbf{S}_{\mu(r_t)}$$

for any sequence $\{r_t\}$ of positive numbers with $+\infty$ limit. Evidently all these sums are equal each other and to the sum μ.

In symbols:

$$\mathbf{S}\mu \equiv \bigcap\, (\mathbf{S}_{\mu(r_t)}\ \{r_t\} \in [0, +\infty], \lim_{t\to\infty} r_t = +\infty)$$

$$\mu \sum_{n=0}^{\infty} a_n = \mu\,(r_t) \sum_{n=0}^{\infty} a_n \text{ for any } \{r_t\} \in [0, +\infty] \text{ with } \lim_{t\to\infty} r_t = +\infty$$

Proof. Immediate consequence from elemental properties of limits.

Remark. The algorithm of

summatory	integral

moments is linear with summation factors

$\mu_{t,n} = \frac{1}{\nu_n} \sum_{\nu=0}^{t} a_\nu \nu^{(n}$ for $t = 0, 1, 2, \ldots;$ $n = 0, 1, 2, \ldots$	$\mu_n(t) = \frac{t}{\mu_n} \int_0^t x(\nu)\, \nu^n d\nu$ for $0 = t \leq +\infty;$ $n = 0, 1, 2, \ldots$ when the associated is an integral function; since then the series ((5′), no. 2) can be added together under the integral sign, because of the uniform convergence in $0 \leq \nu \leq t$ of $a(\nu) \sum_{n=0}^{\infty} a_n \frac{\nu^n}{\mu^n}$

As, after

((3) ; no. 2),	((3′), no. 2),

the algorithm satisfies evidently the edentity principle (Def. 3, no. 2), the foregoing Props. 2, 3 and 4, are included in the (Props. 2, 3 and 4, no. 2) respectively.

In particular, the metric or specific topology for the moment algorithm (Def. 2) coincides, then, in this case, with that one defined (Def. 4, 3, 2) for the linear algorithm, and the same happens with the definition of subordinate algorithms (Def. 3) which coincides with that other given by the linear ones (Def. 5, no. 2).

But, in the continuous case, the moments algorithm us more general because it does not requiere convergence of the associated series in the whole plan, but only the analytical continuation of the associated function along the positive real semiaxis. For this reason we have thought necessary to establish both directly for the moment algorithms.

4. *The Euler's and Borel's associateds*

Def. 1. As it is known the summation algorithm of

Euler	Borel

which will be for short designed by

E	B

is a moments algorithm with the generatrix

$a_t = 2^{-t-1}$ for $t = 0, 1, 2, \ldots$	$a(t) = e^{-t}$ for $0 \leq t < +\infty$

of moments

$n! = \sum_{t=n}^{\infty} 2^{-t-1}\, t^{(n}$	$n! = \int_0^{\infty} e^{-t}\, t^n\, dt$

for $n = 0, 1, 2, \ldots$

The associated is, thus, the

sequence $\{A_t\}$ formed by the values of the polynomials $$A_t = \sum_{n=0}^{t} a_n \frac{t^{(n)}}{n!} \text{ for } t = 0, 1, 2, \ldots$$	function $A(t)$ obtained by analytical continuation along the real positive semi-axis of the holomorph function in 0, defined by the series $$A(t) = \sum_{t=0}^{\infty} a_n \frac{t^n}{n!}$$

This associated has, among other important properties, the following, which we are going to use:

Prop. 1. *The asociated*

E	*B*

of $\sum_{n=1}^{\infty} a_n$ *is*

$\{\Delta A_t\}$	$A'(t)$

Proof. Since

$$\Delta A_t = \left[a_0 + a_1 \binom{t+1}{1} + \ldots + a_i \binom{t+1}{i} + \right.$$ $$\left. + \ldots + a_t \binom{t+1}{t} + a_{t+1} \binom{t+1}{t+1} \right] -$$ $$- \left[a_0 + a_1 \binom{t}{1} + \ldots + a_i \binom{t}{i} + \ldots + a_t \binom{t}{t} \right] =$$ $$= a_1 + a_2 \binom{t}{1} + \ldots + a_t \binom{t}{i-1} + \ldots +$$ $$+ a_t \binom{t}{t-1} + a_{t+1} \binom{t}{t}$$ for $t = 1, 2, \ldots$ and $\Delta A_0 = a_1$	$$A'(t) = a_1 + a_2 \frac{t}{1!} + \ldots + a_n \frac{t^{n-1}}{(n-1)!} + \ldots$$ for $0 \leq t < +\infty$

which is the associated of $\sum_{n=0}^{\infty} a_n$, q. e. d.

Prop. 2. *If* $\sum_{n=0}^{\infty} a'_n$, *with* $$a'_0 = 0 \text{ and } a'_{n+1} = a_n \text{ for } n = 0, 1, 2, \ldots \quad (1)$$ *has the Euler associated* A'_t *with* $$A'_0 = 0 \text{ and } A'_{t+1} = A_t \text{ for } t = 0, 1, 2, \ldots \quad (2)$$ *the* $\sum_{n=0}^{\infty} a''_n$ *with*	Prop. 2'. *The series* $\sum_{n=0}^{\infty} a'_n$ *with* $$a'_0 = 0 \text{ and } a'_{n+1} = a_n \text{ for } n = 0, 1, 2, \ldots \quad (1')$$ *has the Borel associated* $$F(t) = \int_0^t A(v)\, dv$$

$a''_n = a'_n + a'_{n+1}$ for $n = 0, 1, 2, \ldots$
that is
$a''_0 = a_0$ and $a''_n = a_{n-1} + a_n$ for $n = 1, 2, \ldots$
has the Euler associated $\{A_t\}$

Proof. The associated of $\sum_{n=0}^{\infty} a''_n$ is

$$A''_t = a_0 + (a_0 + a_1)\binom{t}{1} + \ldots + (a_{i-1} + a_i)\binom{t}{i} + \ldots + (a_{t-1} + a_t)\binom{t}{t} = a_0\left[1 + \binom{t}{1}\right] +$$

$$+ a_1\left[\binom{t}{1} + \binom{t}{2}\right] + \ldots + a_{i-1}\left[\binom{t}{i-1} + \binom{t}{i}\right] + \ldots + a_{t-1}\left[\binom{t}{t-1} + \binom{t}{t}\right] + a_t\binom{t}{t} = a'_1\binom{t+1}{1} +$$

$$+ a'_2\binom{t+1}{2} + \ldots + a'_i\binom{t+1}{i} + \ldots + a'_t\binom{t+1}{1} + a'_{t+1}\binom{t+1}{t+1} = A'_{t+1} = A_t$$

for $t = 0, 1, 2, \ldots$ q. e. d.

Proof. Since

$$F(t) = a_0 t + a_1 \frac{t^r}{2!} + \ldots + a_{n-1}\frac{t^n}{n!} + \ldots =$$

$$= a'_0 + a'_1 \frac{t}{1!} + \ldots + a'_n \frac{t^n}{n!} + \ldots$$

for $0 \leq t < +\infty$, which is the associated of $\sum_{n=0}^{\infty} a'_n$ q. e. d.

Prop. 3. *If*

$\{A_t\}$ and $\{B_t\}$

Prop. 3′. *If*

$A(t)$ and $B(t)$

are the associated

E | *B*

of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, respectively, and we set

$$c_n = a_n * b_n = \sum_{\nu=0}^{n} a_\nu b_{n-\nu} \text{ for } n = 0, 1, 2, \ldots$$

the associated of $\sum_{n=0}^{\infty} c'_n$ with

$$c'_0 = 0 \text{ and } c'_{n+1} = c_n \text{ for } n = 0, 1, 2, \ldots \tag{3}$$

is the

sequence $\{C'_t\}$ | *function $C(t)$*

defined by

$$C'_0 = 0 \text{ and } C'_{t+1} - C_t = A_t * B_t = \sum_{\nu=0}^{t} A_\nu B_{t-\nu} \quad \text{for } t = 0, 1, 2, \ldots \qquad (4)$$

Proof. By forming the table of differences, we obtain

$$\Delta^n C'_0 = A_0 \Delta^{n-1} B_0 + \Delta A_0 . \Delta^{n-2} B_0 + \Delta^2 A_0 . \Delta^{n-3} B_0 + \ldots + \Delta^{n-3} A_0 . \Delta^2 B_0 + \Delta^{n-2} A_0 . \Delta B_0 + \Delta^{n-1} A_0 . B_0 \qquad (5)$$

for $n = 1, 2, \ldots$

which is easily proved by induction.

In fact, it is evidently

$$\Delta C'_0 = A_0 B_0$$

and being, after (4)

$$C'_t = \sum_{\nu=0}^{t-1} A_\nu B_{t-1-\nu} \quad \text{for } t = 1, 2, \ldots$$

we have successively

$$\Delta C'_t = \sum_{\nu=0}^{t-1} A_\nu . \Delta B_{t-1-\nu} + A_t B_0$$

$$\Delta^2 C'_t = \sum_{\nu=0}^{t-1} A_\nu . \Delta^2 B_{t-1-\nu} + A_t . \Delta B_0 + \Delta A_t . B_0$$

$$\Delta^3 C'_t = \sum_{\nu=0}^{t-1} A_\nu . \Delta^3 B_{t-1-\nu} + A_t . \Delta^2 B_0 + \Delta A_t . \Delta B_0 + \Delta^2 A_t . B_0$$

. . . .

$$\Delta^n C'_t = \sum_{\nu=0}^{t-1} A_\nu . \Delta^n B_{t-1-\nu} + A_t . \Delta^{n-1} B_0 + \Delta A_t . \Delta^{n-2} B_0 + \Delta^2 A_t . \Delta^{n-3} B_0 + \ldots + \Delta^{n-3} A_t . \Delta^2 B_0 + \Delta^{n-2} A_t . \Delta B_0 + \Delta^{n-1} A_t . B_0$$

for $t = 1, 2, 3, \ldots$ Specially, for $t = 1$, the last one yields

$$\Delta^n C'_1 = A_0 . \Delta^n B_0 + A_1 . \Delta^{n-1} B_0 + \Delta A_1 . \Delta^{n-2} B_0 + \Delta^2 A_1 . \Delta^{n-3} B_0 + \ldots + \Delta^{n-3} A_1 . \Delta^2 B_0 + \Delta^{n-2} A_1 . \Delta B_0 + \Delta^{n-1} A_1 . B_0$$

$$C(t) = A * B = \int_0^t A(\nu)\, B(t-\nu)\, d\nu \quad \text{for } 0 \le t \le +\infty \qquad (4')$$

Proof. It is deduced from [5, 11-12] by virtue of Prop. 2'; since if $w(t)$ is the associated of $\sum_{n=0}^{\infty} c_n$, that of $\sum_{n=0}^{\infty} c'_0$ is $\int_0^t W(\nu)\, d\nu$ according to Prop. 2'; and by virtue of [5, 11-12] we have

$$\int_0^t W(\nu)\, d\nu = C(t)$$

whence, after (5):

$$\Delta^{n+1} C'_0 = \Delta^n C'_1 - \Delta^n C_0' = A_0 . \Delta^n B_0 + \\ + \Delta A_0 \Delta^{n-1} B_0 + \Delta^2 A_0 \Delta^{n-2} B_0 + \\ + \Delta^3 A_0 . \Delta^{n-3} B_0 + \ldots + \Delta^{n-2} A_0 . \Delta^2 B_0 + \\ + \Delta^{n-1} A_0 . \Delta B_0 + \Delta^n A_0 . B_0$$

which is just the (5) for $n + 1$.

But being $\{A_t\}$ and $\}B_t\}$ the associateds of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, respectively, we have ((3), no. 3):

$a_n = \Delta^n A_0$ and $b_n = \Delta^n B_0$ for $n = 0, 1, 2, \ldots$; then, after (5), it is

$$\Delta^n C'_0 = a_0 b_{n-1} + a_1 b_{n-2} + \ldots + a_{n-1} b_0 = \\ = c_{n-1} = c'_n$$

for $n = 1, 2, \ldots$; and as furthermore

$$C'_0 = 0 \quad \text{and} \quad c'_0 = 0$$

it results actually, after ((3), no. 3) that $\{C'_0\}$ is the associated of $\sum_{n=0}^{\infty} c'_n$, q. e. d.

Remark. For the series $\sum_{n=0}^{\infty} a_n$ with

$$l = \lim_{n\to\infty} |a_n|^{1/n} < \infty,$$

the conclusion of this proposition becomes easily from the identity:

$$\sum_{n=0}^{\infty} a_n z^{n+1} = \sum_{t=0}^{\infty} A_t \left(\frac{z}{z+1}\right)^{t+1}$$

which remains valid [7, 7 and 179] for $|z| < 1/l$. But although these are the only summable series E, the restriction is artificial and needless for the validity of the result.

Prop. 4. *If $\{A_t\}$ and $\{B_t\}$ are the associated of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, respectively, and* *we set* $c_n = a_n * b_n$ *for* $n = 0, 1, 2, \ldots$ *the associated of* $\sum_{n=0}^{\infty} c''_n$ *with*

$c''_0 = c_0$ *and* $c''_n = c_{n-1} + c_n$ *for* $n = 1, 2, \ldots$ *is* $\{A_t * B_t\}$.

Proof. It suffices to apply Prop. 2, to the $\sum_{n=0}^{\infty} c'_n$ and $\{C'_t\}$ defined in Prop. 3 above; since conditions (3) and (4) of the said Prop. 3 are exactly (1) and (2) of Prop. 2; and furthermore $\sum_{n=0}^{\infty} c'_n$ has the associated $\{C'_t\}$ after this Prop. 3, which is the other assumption in Prop. 2, that assures that the associated of $\sum_{n=0}^{\infty} a''_n$ is $C_t = A_t * B_t$ for $t = 0, 1, 2, \ldots$

5. *Identity of the sums with the Euler's or Borel's summation algorithms, and with a weaker one satisfying the formal laws of no.* **1**.

It is expressed by the following theorem:

Theor. 1. *Let* σ *be a summation algorithm weaker (Def.* 3, *no.* 1*) than that of*

Euler E | *Borel* B

If σ *satisfies the formal laws a', b', and c' or*

d', | d'_1,

satisfying besides the e' and f' with the specific topology for the algorithm

E, | B,

we have

$$\sigma \sum_{n=0}^{\infty} a_n = E \sum_{n=0}^{\infty} a_n \qquad (1)$$ | $$\sigma \sum_{n=0}^{\infty} a_n = B \sum_{n=0}^{\infty} a_n \qquad (1')$$

for the whole summable series σ:

$$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma$$

Proof. Since

$\mathbf{S}_\sigma \subset \mathbf{S}_E$, | $\mathbf{S}_\sigma \subset \mathbf{S}_B$,

every summable series σ has an associated

$\{A_t\}$ in E, | $A(t)$ in B,

i. e., the vectorial space

$\mathbf{u}_1$, | $\mathbf{U}_1$,

formed by the products

$\{2^{-t} A_t\}$ of 2^{-t}	$e^{-t} A(t)$ of e^{-t}

of the associated

E	B

of the series of $\mathbf{S}\sigma$, is contained in the vectorial space

$\mathbf{u}_0$	$\mathbf{U}_0$

of the products of

2^{-t}	e^{-t}

for the associateds of all the summable series

E,	B,

which in turn, belongs to the space (Def. 1, no. 1) :

$\mathbf{u}$;	$\mathbf{U}$;

i. e., we have

$\mathbf{u}_1 \subset \mathbf{u}_0 \subset \mathbf{u}$ (2)	$\mathbf{U}_1 \subset \mathbf{U}_0 \subset \mathbf{U}$ (2′)

being

$\mathbf{u}_1 \sim \mathbf{S}_\sigma$ (3)	$\mathbf{U}_1 \sim \mathbf{S}_\sigma$ (3′)

that is, coordinable with $\mathbf{S}\sigma$

But, by virtue of e', if

$$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma,$$

it is also

$$\sum_{n=0}^{\infty} a_n x^n \in \mathbf{S}_\sigma \text{ for } x \in \,]0, 1[,$$

then, the sum

$$\sigma \sum_{n=0}^{\infty} a_n x^n \text{ for } x \in \,]0, 1[$$

is a functional defined in $\mathbf{S}\sigma$, at least, and depending on a parameter $x \varepsilon \,]0, 1[$ that is, according to

(3)	(3′)

a functional defined, at least, in

$\mathbf{u}_1$	$\mathbf{U}_1$

and depending on the parameter $x \varepsilon \,]0, 1[$ or on the

$z \in \,]0, 1[$	$z \in \,]0, +\infty[$

writing

$x = \dfrac{z}{2-z}$	$x = \dfrac{1}{z+1}$

That is, if we define:

$$\mathcal{F}_z(2^{-t}A_t) = \frac{2}{2-z} \cdot \sigma \sum_{n=0}^{\infty} a_n \left(\frac{z}{2-z}\right)^n \quad (4) \qquad \Bigg| \qquad \mathcal{F}_z\left[e^{-t}A(t)\right] = \frac{1}{z+1} \cdot \sigma \sum_{n=0}^{\infty} \frac{a_n}{(z+1)^n} \quad (4')$$

for the associated

$\{A_t\}$ in E | $A(t)$ in B

of each series

$$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma,$$

we have a functional defined, at least, in

$\mathbf{u}_1$, | $\mathbf{U}_1$,

and depending on the parameter

$z \in \,]0,\ 1[$ | $z \in \,]0,\ +\infty[$

The functional $\mathcal{F}_z$ is furthermore defined through

(4), | (4′),

not only for

$\mathbf{u}_1$, | $\mathbf{U}_1$,

but for the vectorial space

$\mathbf{u}_2 \supset \mathbf{u}_1$ (5) | $\mathbf{U}_2 \supset \mathbf{U}_1$ (5′)

(wider in fact, after e') formed by the products

$\{2^{-t}A_t\}$ of 2^{-t} | $e^{-t}A(t)$ of e^{-t}

by the associated of the series $\sum\limits_{n=0}^{\infty} a_n$ such that

$$\sum_{n=0}^{\infty} a_n x^n \in \mathbf{S}_\sigma \text{ for } x \in \,]0,\ 1[$$

that is

$$\sum_{n=0}^{\infty} a_n \left(\frac{z}{2-z}\right)^n \in \mathbf{S}_\sigma \text{ for } z \in \,]0,\ 1[\qquad \Bigg| \qquad \sum_{n=0}^{\infty} \frac{a_n}{(z+1)^n} \in \mathbf{S}_\sigma \text{ for } z \in \,]0,\ +\infty[;$$

among which are obviously included, all the series $\sum\limits_{n=0}^{\infty} a_n$ with $\overline{\lim\limits_{n\to\infty}}\, |a_n|^{1/n} \leq 1$, because of the permanency of the algorithm σ.

Furthermore, since σ verifies the laws a', b' and f' with the specific topology of this functional

E | B

this functional $\mathcal{F}_z$ is for each

$z \in \,]0,\ 1[$ | $z \in \,]0,\ +\infty[$

distributive and continuous, i. e., linear [1, 23] in

$\mathbf{u}_1$	$\mathbf{U}_1$

with the subordinate topology, according to

(2), in $\mathbf{u}_1$	(2′), in $\mathbf{U}_1$

by the topology of

$\mathbf{u}_0$,	$\mathbf{u}_0$,

which is in turn, after (Def. 2, no. 3) that one defined by the metric of

$\mathbf{u}$	$\mathbf{U}$

But

$\mathbf{u}_1$	$\mathbf{U}_1$

being, after

(2)	(2′)

a vectorial subspace of the metrical space

$\mathbf{u}$,	$\mathbf{U}$,

the functional $\mathcal{F}_z$ can be extended [1, Theor. 2, 55] to

$\mathbf{u}$	$\mathbf{U}$

and we have a functional continuous in

$\mathbf{u}$	$\mathbf{U}$

and depending on the parameter

$z \in]0, 1[$	$z \in]0, +\infty[$

wich we will design with the same notation

(4)	(4′)

Now, applying successively

a', b', (4), (Prop. 1, no. 3), c'_1 and (4),	a', (4′), b' (Prop. 1, no. 4), c'_1 and (4′)

we have:

$$z\, \mathcal{F}_z \left[\Delta \left(2^{-t} A_t\right)\right] = z\, \mathcal{F}_z \left(2^{-t-1} \Delta A_t\right) - z\, \mathcal{F}_z \left(2^{-t-1} A_t\right) = \frac{z}{2} \mathcal{F}_z \left(2^{-t} \Delta A_t\right) - \frac{z}{2} \mathcal{F}_z\left(2^{-t} A_t\right) = \frac{z}{2-z} \cdot \sigma \sum_{n=1}^{\infty} a_n \left(\frac{z}{2-z}\right)^{n-1} - \frac{z}{2} \mathcal{F}_z \left(2^{-t} A_t\right) = \sigma \sum_{n=1}^{\infty} a_n \left(\frac{z}{2-z}\right)^{n} - \frac{z}{2} \mathcal{F}_z \left(2^{-t} A_t\right) = \sigma \sum_{n=0}^{\infty} a_n \left(\frac{z}{2-z}\right)^{n} - a_0 -$$

$$\mathcal{F}_z \left\{ \frac{d}{dt} \left[e^{-t} A(t) \right] \right\} = \mathcal{F}_z \left[e^{-t} A'(t)\right] - \mathcal{F}_z \left[e^{-t} A(t)\right] = \frac{1}{z+1} \cdot \sigma \sum_{n=1}^{\infty} \frac{a_n}{(z+1)^{n-1}} - \mathcal{F}_z \left[e^{-t} A(t)\right] = \sigma \sum_{n=1}^{\infty} \frac{a_n}{(z+1)^n} - \mathcal{F}_z\left[e^{-t} A(t)\right] = \sigma \sum_{n=0}^{\infty} \frac{a_n}{(z+1)^n} - a_0 - \mathcal{F}_z \left[e^{-t} A(t)\right] = (z+1)\, \mathcal{F}_z \left[e^{-t} A(t)\right] - a_0 - \mathcal{F}_z \left[e^{-t} A(t)\right] =$$

$$-\frac{z}{2}\mathcal{F}_z(2^{-t}A_t) = \frac{2-z}{2}\mathcal{F}_z(2^{-t}A_t) - a_0 -$$

$$-\frac{z}{2}\mathcal{F}_z(2^{-t}A_t) = (1-z)\mathcal{F}_z(2^{-t}A_t) -$$

$$-[2^{-t}A_t]_{t=0}$$

which is the incrementation law for

$$\{2^{-t}A_t\} \in \mathbf{u}_2$$

$$= z\,\mathcal{F}_z[e^{-t}A(t)] - [e^{-t}A(t)]_{t=0}$$

which is the derivation law for

for $[e^{-t}A(t)] \in \mathbf{U}_2$

On the other hand, although the

constant sequence $\xi_t = 1$ for $t = 0, 1, 2, \ldots$ | constant function $\xi(t) = 1$ for $0 \leq t \leq +\infty$

does not belong to

$\mathbf{u}$, | $\mathbf{U}$,

and, therefore, neither

(2), to $\mathbf{u}_1$ | (2′), to $\mathbf{U}_1$

it surely belongs to

$$\mathbf{u}_2 \supset \mathbf{u}_1\,; \tag{5}$$

$$\mathbf{U}_2 \supset \mathbf{U}_1\,; \tag{5'}$$

since

$\{2^t\}$ | e^t

is, according to

((3), no. 3) | ((3′), no. 3)

the associated

E | B

of the divergent series $\sum_{n=0}^{\infty} 1 = \infty$, but of radius 1, and we have, because of the permanency of σ

$$\mathcal{F}_z(\xi_t) = \frac{2}{2-z}\cdot\sigma\sum_{n=0}^{\infty}\left(\frac{z}{2-z}\right)^n =$$

$$= \frac{2}{2-z}\,\frac{1}{1-\dfrac{z}{2-z}} = \frac{1}{1-z} \text{ for } z \in \,]0, 1[$$

$$\mathcal{F}_z(\xi) = \frac{1}{z+1}\cdot\sigma\sum_{n=0}^{\infty}\frac{1}{(z+1)^n} =$$

$$= \frac{1}{z+1}\,\frac{1}{1-\dfrac{1}{z+1}} = \frac{1}{z} \text{ for } z \in \,]0, +\infty[,$$

which is the initial condition

((14), Theor. 2, no. 3, § 1) | ((14′), Theor. 2, no. 3, § 1)

The sequence $\{\xi_t^{(0)}\}$ with

$\xi_t^{(0)} = 1$ and $\xi_t^{(0)} = 0$ for $t = 1, 2, \ldots$

belongs evidently to $\mathbf{u}$, but it does to $\mathbf{u}_2$; since the sequence $\{2^t\,\xi_t^{(0)}\}$ is, according to

((3), no. 3), the E-associated of the non convergent series $\sum_{n=0}^{\infty}(-1)^n$, with radius equal 1 too, and we have

$$\mathcal{F}_z(\xi_t^{(0)}) = \frac{2}{2-z}\cdot\sigma\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2-z}\right)^n =$$

$$= \frac{2}{2-z}\,\frac{1}{1+\dfrac{z}{2-z}} = 1$$

which is the second initial condition ((13), Theor. 2, no. 3, § 1).

Let us see now the convolution law. If

$\{A_t\}$ and $\{B_t\}$ | $A(t)$ and $B(t)$

are, respectively, the associated of

$$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma \text{ and } \sum_{n=0}^{\infty} b_n \in \mathbf{S}_\sigma$$

and we write

$$c_n = a_n * b_n = \sum_{\nu=0}^{n} a_\nu\, b_{n-\nu} \text{ for } n = 0,\ 1,\ 2,\ \dots$$

by applying successively

(4) (Prop. 4, no. 4), b', d' and (4) | (4), (Prop. 3′, no. 4), a', b' d'_1 and (4′)

we have

$$\mathcal{F}_z[(2^{-t}A_t)*(2^{-t}B_t)] = \mathcal{F}_z[2^{-t}(A_t*B_t)] =$$

$$= \frac{2}{2-z}\cdot\sigma\left[c_0+(c_0+c_1)\frac{z}{2-z}+\dots+\right.$$

$$\left.+(c_{n-1}+c_n)\left(\frac{z}{2-z}\right)^n+\dots\right] =$$

$$= \frac{2}{2-z}\cdot\sigma\left[c_0+c_1\frac{z}{2-z}+\dots+c_n\left(\frac{z}{2-z}\right)^n+\dots\right] +$$

$$+ \frac{2}{2-z}\cdot\sigma\left[c_0\frac{z}{2-z}+c_1\left(\frac{z}{2-z}\right)^2+\dots+\right.$$

$$\left.+c_n\left(\frac{z}{2-z}\right)^{n+1}+\dots\right] = \frac{2}{2-z}\cdot\sigma\left[c_0+c_1\frac{z}{2-z}+\right.$$

$$\mathcal{F}_z\{[e^{-t}A(t)]*[e^{-t}B(t)]\} = \mathcal{F}_z\{e^{-t}[A(t)*$$

$$*B(t)]\} = \frac{1}{z+1}\cdot\sigma\left[0+\frac{c_0}{z+1}+\frac{c_1}{(z+1)^2}+\right.$$

$$\left.+\dots+\frac{c_n}{(z+1)^{n+1}}+\dots\right] =$$

$$= \frac{1}{(z+1)^2}\cdot\sigma\left[0+c_0+\frac{c_1}{z+1}+\dots+\right.$$

$$\left.+\frac{c_n}{(z+1)^n}+\dots\right] = \frac{1}{(z+1)^2}\cdot\sigma\left[0+a_0b_0+\right.$$

$$\left.+\frac{a_1*b_1}{z+1}+\dots+\frac{a_n*b_n}{(z+1)^n}+\dots\right] =$$

$$+ \dots + c_n \left(\frac{z}{2-z}\right)^n + \dots \Big] +$$

$$+ \frac{2}{2-z} \frac{z}{2-z} \cdot \sigma \left[c_0 + c_1 \frac{z}{2-z} + \dots + \right.$$

$$\left. + c_n \left(\frac{z}{2-z}\right)^n + \dots \right] = \left(\frac{2}{2-z}\right)^2 \cdot \sigma \left[c_0 + c_1 \frac{z}{2-z} + \right.$$

$$\left. + \dots + c_n \left(\frac{z}{2-z}\right)^n + \dots \right] = \left(\frac{2}{2-z}\right)^2 \cdot \sigma \left[a_0 b_0 + \right.$$

$$\left. + (a_1 * b_1) \frac{z}{2-z} + \dots + (a_n * b_n) \left(\frac{z}{2-z}\right)^n + \dots \right] =$$

$$= \left(\frac{2}{2-z}\right)^2 : \sigma \left\{ a_0 b_0 + \left[\left(a_1 \frac{z}{2-z}\right) * \left(b_1 \frac{z}{2-z}\right)\right] + \right.$$

$$\left. + \dots + \left(\left[a_n \left(\frac{z}{2-z}\right)^n\right] * \left[b_n \left(\frac{z}{2-z}\right)^n\right]\right) + \dots \right\} =$$

$$= \frac{2}{2-z} \cdot \sigma \left[a_0 + a_1 \frac{z}{2-z} + \dots + a_n \left(\frac{z}{2-z}\right)^n + \right.$$

$$\left. + \dots \right] \cdot \frac{2}{2-z} \cdot \sigma \left[b_0 + b_1 \frac{z}{2-z} + \dots + \right.$$

$$\left. + b_n \left(\frac{z}{2-z}\right)^n + \dots \right] = \mathcal{F}_z \, [2^{-t} A_t] \cdot \mathcal{F}_z \, [2^{-t} B_t]$$

$$= \frac{1}{(z+1)^2} \sigma \left[0 + a_0 b_0 + \left[\left(\frac{a_1}{z+1}\right) * \left(\frac{b_1}{z+1}\right)\right] + \right.$$

$$\left. + \dots + \left[\frac{a_n}{(z+1)^n} * \frac{b_n}{(z+1)^n}\right] + \dots \right\} =$$

$$= \frac{1}{z+1} \cdot \sigma \left[a_0 + \frac{a_1}{z+1} + \dots + \frac{a_n}{(z+1)^n} + \dots \right] .$$

$$\cdot \frac{1}{z+1} \cdot \sigma \left[b_0 + \frac{b_1}{z+1} + \dots + \frac{b_n}{(z+1)^n} + \dots \right] =$$

$$= \mathcal{F}_z \, [e^{-t} A_t(t)] \cdot \mathcal{F}_z \, [e^{-t} B(t)]$$

for every pair:

$\{2^{-t} A_t\} \in \mathbf{u}_2$ and $\{2^{-t} B_t\} \in \mathbf{u}_2$ | $[e^{-t} A(t)] \in \mathbf{U}_2$ and $[e^{-t} B(t)] \in \mathbf{U}_2$

It does not seem easy to prove that these

incrementation | derivation

or convolution laws hold in the whole space

u | **U**

But this is not either necessary in order to apply the (Theor. 1 and 2, no. 3, § 1) of characterization; since after (Theor. 3, no. 3, § 1) we can take in said (Theor. 1 and 2, n.º 3, § 1), as a complete sequence in the space the

$\varphi_{n,t} = 2^{-t} t^{(n+1}$ for $t = 0, 1, 2, \dots$ | $\varphi_n(t) = e^{-t} t^n$ for $0 \leq t \leq +\infty$

and $n = 0, 1, 2, \dots$

which not only belonge to $\mathbf{u}_2$, but also tu $\mathbf{u}_1$ | whose product by e^{-at} belongs surely to $\mathbf{U}_2$, if the strictly positive constant is strictly greater than 1, i. e., $a \in]0, 1[$

then, the

sequence $\{t^{(n+1}\}$	function $e^{-at}\, t^n$

for each $n = 0, 1, 2, \ldots$ is the associated

E	B

of a series $\sum_{i=0}^{\infty} \omega_i^{(n)}$ with the terms

$\omega_{n+1}^{(n)} = (n+1)!$ and $\omega_i^{(n)} = 0$ for $i \neq n+1$ which evidently converge

$$\sum_{t=0}^{\infty} \omega_i^{(n)} < \infty \text{ for } n = 0, 1, 2, \ldots$$

We saw before that the sequences $\{\xi_t\}$ and $\{\xi_t^{(0)}\}$ belonged to $\mathbf{u}_2$, and thereafter, also the sequence $\{\xi_t - \xi_t^{(0)}\}$. Also the $\{\xi'_t\} \in \mathbf{u}_2$ ((22), no. 3, § 1) since $\{2^t\, \xi'_t\}$ is, according to ((3), no. 3) the E- associated of the series $\sum_{n=0}^{\infty} (-1)^{n+1}\, 2n$ with convergence radius equal one.

On the other hand, being $\{2^t\, \varphi_{n,t}\}$ the associated of $\sum_{i=0}^{\infty} \omega_i^{(n)}$ and the difference

$$\Delta_t\, 2^t\, \varphi_{n,t} = 2^t\, \varphi_{n,t} + 2^{t+1}\, \Delta_t\, \varphi_{n,t},$$

for $t = 0, 1, 2, \ldots$ and each $n = 0, 1, 2, \ldots$, the associated of $\sum_{i=1}^{\infty} \omega_i^{(n)}$ (Prop. 1, n.º 4 according to a, b and c_1, $\{2^t\, \Delta_t\, \varphi_{n,t}\}$ the associated of (*)

$$\frac{1}{2}\left[\sum_{i=1}^{\infty} \omega_i^{(n)} - \sum_{i=0}^{\infty} \omega_i^{(n)}\right] = -\,\omega_0^{(n)}/2 = 0$$
for each $n = 0, 1, \ldots$

$$\omega_i^{(n)} = 0 \text{ for } 0 \leq i \leq n-1 \quad (\text{if } n > 0)$$

$$\omega_i^{(n)} = \frac{i!}{(i-n)!}(-a)^{i-n} \text{ for } i \geq n \geq 0.$$

which actually has a convergence radius $a \in \,]0, 1[$.

$$\lim_{i\to\infty} |\omega_i^{(n)}|^{1/i} = a \text{ for each } n = 0, 1, 2, \ldots$$

Specially, for $n = 0$, it results that the function $e^{-at} \in \mathbf{U}_2$ and, therefore, the convolution law holds for the products (19′) of (Theor. 2, no. 3, § 1).

(*) This can be so proved directly:

$$\Delta\varphi_{n,t} = 2^{-t-1}\,(n+1)\, t^{(n} - 2^{-t-1}\,(t+1)^{(n}$$

$$2^t\, \Delta\, \varphi_{n,t} = \frac{1}{2}\left[(n+1)\, t^{(n} - (t+1)^{(n}\right]$$

which, up to the factor $\frac{1}{2}$, is the associated to the difference between the two series $\sum_{i=1}^{\infty} \omega_i^{(n)}$ and $\sum_{i=1}^{\infty} \omega_i^{(n)}$, whose terms are all zero, excepted the n-th of the former and the $(n+1)$-*th* of the second, both equal to $(n+1)!$

then

$$\{\Delta_t \varphi_{n,t}\} \in \mathbf{u}_1 \subset \mathbf{u}_2$$

Furthermore, according to (Prop. 2, no. 4), since $\sum_{i=0}^{\infty} \omega_i^{(n)}$ with $\omega_0^{(n)} = 0$ has the a $\{2^t \varphi_{n,t}\}$ with $\varphi_{n,0} = 0$ for each $n = 0, 1, 2, \ldots$, the $\{2^{t+1} \varphi_{n,t+1}\}$ is the associated of $\sum_{i=0}^{\infty} \zeta_i^{(n)} < \infty$ with $\zeta_i^{(n)} = \omega_i^{(n)} + \omega_{i+1}^{(n)}$ for $i = 0, 1, 2, \ldots$ and each $n = 0, 1, 2, \ldots$ that is, $\{2^t \varphi_{n,t+1}\}$ is the associated of $\frac{1}{2}\sum_{i=0}^{\infty} \zeta_i{}^{(n)} < \infty$ for each $n = 0, 1, 2, \ldots$; therefore, also

$$\{\varphi_{n,t+1}\} \in \mathbf{u}_1 \subset \mathbf{u}_2$$

The convolution law is then verified for all the products (18), (19) and (20) of (Theor. 2, no. 3, § 1)

Once thus proved all the hypotheses of the characterization theorems (Theor. 1, 2, 3, no. 3, § 1) we have:

$$\mathcal{F}_z[2^{-t} A_t] = \sum_{t=0}^{\infty} 2^{-t} A_t z^t \text{ for } \{2^{-t} A_t\} \in \mathbf{u}_1$$

and $z \in \,]0, 1[$

or according to (4);

$$\sigma \sum_{n=0}^{\infty} a_n \left(\frac{z}{2-z}\right)^n = \frac{2-z}{z} \sum_{t=0}^{\infty} 2^{-t} A_t z^t$$

for $z \in \,]0, 1[$

$$\mathcal{F}_z[e^{-t} A(t)] = \int_0^{\infty} e^{-t} A(t) e^{-tz} dt \text{ for}$$

$[e^{-t}A(t)] \in \mathbf{U}_1$ and $z \in \,]0, +\infty[$

(4');

$$\sigma \sum_{n=0}^{\infty} \frac{a_n}{(z+1)^n} = (z+1) \int_0^{\infty} e^{-t} A(t) e^{-tz} dt$$

for $z \in \,]0, +\infty[$

whatever the σ-summable series $\sum_{n=0}^{\infty} a_n \in \mathbf{S}_\sigma$, may be. Whence it follows the conclusion

(1), for $z \to 1-$ | (1'), for $z \to 0+$

by applying to the first member the law e', and to the second the classical Abel's theorem

for powers' series | for Laplace's transformation

The theorem is thus proved.

6. *Algorithms of convergence of moments.*

Def. 1. We call

summatory- | *Stieltjes-*

algorithm of convergence of moments, of *p-type*, to a permanent linear algorithm (Def. 1, no. 2), whose convergence factors are

$$\lambda_{t,n} = \alpha_t \frac{t^{(n+p)}}{\nu_{n+p}} \text{ for } t = 0, 1, 2, \ldots \quad (1)$$

$$\lambda_n(t) = \alpha(t) \frac{t^{n+p}}{\mu_{n+p}} \text{ for } 0 \leq t < +\infty \quad (1')$$

and $n = 0, 1, 2, \ldots$, where $p \geq 0$ is a fixed integer independent from t and n, and the

$\{\alpha_t\}$ any arbitrary sequence

$\alpha(t)$ a function of limited variation in each finite interval $[0, T]$,

such that

$$\nu_n = -\sum_{t=0}^{\infty} t^{(n} \Delta\alpha_t < \infty$$

$$\mu_n = -\int_0^{\infty} t^n \, d\,\alpha(t)$$

for $n = 0, 1, 2, \ldots$; or more general to the one defining the limit for a sequence $\{s_n\}$ as follows:

$$\nu \lim s_n = \sum_{t=0}^{\infty} \alpha(t) \sum_{n=0}^{t+p} s_n \frac{t^{(n+p)}}{\nu_{n+p}}$$

$$\mu \lim s_n = \int_0^{\infty} \alpha(t) \sum_{n=0}^{\infty} s_n \frac{t^{n+p}}{\mu_{n+p}} dt$$

where

$$A_t = \sum_{n=0}^{t+p} s_n \frac{t^{(n+p)}}{\nu_{n+p}} \text{ for } t = p, p+1$$

is a polynomial whose values make a sequence

$$A(t) = \sum_{n=0}^{\infty} s_n \frac{t^{n+p}}{\mu_{n+p}} \text{ for } 0 \leq t < +\infty$$

is the function obtained by analytical continuation along the positive real semiaxis of the function holomorph in 0 defined by the series

which is called the *associated for the sequence* $\{s_n\}$ [17, 18] in the convergence algorithm.

Remark. The proof of the permanence conditions [12, 195] requires a detailed study of the

sequence of polynomials $\left\{ \sum_{t=0}^{n} \frac{t^{(n)}}{\nu_n} \right\}$

integral function $\sum_{n=0}^{\infty} \frac{t^n}{\mu_n}$

as a generalization of the

$\{z^t\}$

exponential and the E_α of Mittag-Leffler functions.

Def. 2′. We will call *regular sequence* with respect to a μ-algorithm of Stieltjes moments $\{\mu_n\}$ to any increasing sequence with limit $+\infty$, and strictly greater than 1, in wich, furthermore, the generatrix $\alpha(t)$ remains structly positive, that is

$$\lim_{t\to\infty} r_t = +\infty, \; 1 < r_t \leq r_{t+1} \text{ and } \alpha(r_t) > 0 \quad \text{for } t = 0, 1, 2, \ldots \quad (2')$$

With this terminology we can briefly state the following

Prop. 1. *Any linear convergence algorithm of summatory moments* $\{v_n\}$ *is reversible* [1, 90].

Proof. In fact it is normal [9, 600]

Prop. 1′. *Any algorithm subordinate to a linear* μ*-algorithm of integral moments through a regular sequence is reversible.*

Proof. This means [1, 90] that for any convergent sequence $\{\eta_t\}$, there exists another $\{s_n\}$, convergent or not, but such that

$$\sum_{n=0}^{\infty} \lambda_n(r_t)\, s_n = \eta_t \text{ for } t = 0, 1, 2, \ldots \quad (3')$$

or after (1′)

$$\sum_{n=0}^{\infty} \frac{s_n}{\mu_{n+p}} r_t^n = \frac{\eta_t}{\alpha(r_t) r_t^p} < \infty \text{ for } t = 0, 1, 2, \ldots \quad (4')$$

Now then, [1, 76], the necessary and suficient condition for the system of infinite linear equations

$$\sum_{n=0}^{\infty} r_t^n\, z_n = \frac{\eta_t}{\alpha(r_t) r_t^p} \text{ for } t = 0, 1, 2, \ldots \quad (5')$$

to have a solution $\{z_n\}$ such that

$$\sum_{n=0}^{\infty} |z_n| = M < +\infty \quad (6')$$

is that for any finite sequence of numbers $h_0, h_1, \ldots, h_m$, is be verified

$$\left| \sum_{t=0}^{m} h_t \frac{\eta_t}{\alpha(r_t) r_t^p} \right| \leq M. \sup_{0 \leq n < +\infty} \left| \sum_{t=0}^{m} h_t r_t^n \right|$$

$$\text{for } m = 0, 1, 2, \ldots \quad (7')$$

But being, by assumption, $r_t > 1$ for $t = 0, 1, 2, \ldots$ this upper bound of the second member is surely $+\infty$; therefore condition (7′) is verified, whatever M may be, that is, the system (5′) has a solution $\{z_n\}$ which verifies (6′) for any prefixed constant M, and therefrom, the solution of of (4′) follows by setting

$$s_n = z_n\, \mu_{n+p} \text{ for } n = 0, 1, 2, \ldots$$

Prop. 2. *A convergence algorithm of p-type of summatory moments* $\{v_n\}$

Prop. 2′. *A convergence algorithm subordinate to a linear* μ*-algorithm of Stieltjes moments* $\{\mu_n\}$ *through a regular sequence* $\{r_t\}$ *(Def. 2′).*

is perfect if the sequence

$$\{a_t\ t^{(n+p}\} \qquad \Big| \qquad \{\alpha\ (r_t)\ r_t^{n+p}\}$$

being $p = 0$ a fixed integer independent from t and n, is very weakly complete with respect to **u**. (Def. 5, no. 2).

Proof. If, for a sequence $\{s_t\}$ holds:

$$\sum_{t=0}^{\infty} \lambda_{t,\,n}\ s_t = 0 \qquad \Big| \qquad \sum_{t=0}^{\infty} \lambda_n\ (r_t)\ s_t = 0$$

for $n = 0, 1, 2, \ldots$, we have evidently, according to

$$(1) \quad \frac{1}{\nu_{n+p}} \sum_{t=0}^{\infty} a_t\ t^{(n+p}\ s_t = 0 \qquad \Big| \qquad (1') \quad \frac{1}{\mu_{n-p}} \sum_{t=0}^{\infty} \alpha\ (r_t)\ r_t^{n+p}\ s_t = 0$$

or

$$\sum_{t=0}^{\infty} a_t\ t^{(n+p}\ s_t = 0 \qquad \Big| \qquad \sum_{t=0}^{\infty} \alpha\ (r_t)\ r_t^{n+p}\ s_t = 0$$

for $n = 0, 1, 2, \ldots$

But, being the sequence

$$\{a_t\ t^{(n+p}\} \qquad \Big| \qquad \{\alpha\ (r_t)\ r_t^{n+p}\}$$

very weakly complete with respect **u**, if $\{s_t\} \in$ **u**, it will be (Def. 5, no. 2, S 1):

$$s_t = 0 \text{ for } t = 0, 1, 2, \ldots$$

As, on the other hand, according to Prop. 1 above, the algorithm is reversible, it shall be perfect [1, 90], q. e. d.

Theor. 1. *Let be* (Def. 1, no. 6)

$$\nu \qquad \Big| \qquad \mu$$

a linear convergence algorithm of p-type of moments

$$\nu_n = \sum_{t=0}^{n} a_t\ t^{(n} < \infty \qquad \Big| \qquad \mu_n = \int_0^{\infty} \alpha\ (t)\ t^n\ dt < \infty$$

for $n = 0, 1, 2, \ldots$ such that

| *for any regular sequence $\{r_t\}$ (Def. 2′),*

the sequence

$$\{a_t\ t^{(n+p}\}, \qquad \Big| \qquad \{\alpha\ (r_t)\ r_t^{n+p}\}$$

where $p \geq 0$ is an integer independent from t and n, is very weakly complete with respect to **u** (Def. 5, no. 2. § 1).

If λ is a permanent linear convergence algorithm, with matrix or continuous (Def. 1, no. 2), stronger (Def. 3, no. 1) than the

ν | *set or union (Def. 4, no. 1) of the subordinate algoritms (Def. 5, no. 2) to the μ through some regular sequences (Def. 2′).*

we have

$\nu \lim s_n = \lambda \lim s_n$ (9)	$\mu \lim s_n = \lambda \lim s_n$ (9′)

for every sequence

$\{s_n\} \epsilon \mathbf{s}_\lambda$	$\{s_n\} \epsilon \mathbf{s}_\mu$
λ*-convergent*	μ*-convergent*

Proof. It is a consequence of Banach fundamental [Theorem 1, Theor, 12, 95] by virtue of

Prop. 2	Prop. 2′

In fact, if λ comes from a matrix, we have assumption,

$\nu \subset \lambda$ (10)	according to (Defs. 3 and 4, no. 1) $\mu\,(r_t) \subset \lambda$ for every regular sequence $\{r_t\}$; (10′)

and, if λ is continuous, for every sequence $\{r'_t\} \epsilon [0, +\infty]$ with limit $+\infty$, we have, according to

(10)	(10′)

(Cor. 1). Prop. 4, no. 2) and (Prop. 1, no. 1)

$\nu \subset \lambda\,(r'_t)$	$\mu\,(r_t) \subset \lambda\,(r'_t)$

But

ν	$\mu\,(r_t)$

being perfect, according to

Prop. 2	Prop. 2′

for every sequence

$\{s_n\} \epsilon \mathbf{s}_\nu$	$\{s_n\} \epsilon \mathbf{s}_\mu \subset \mathbf{s}_{\mu(r_t)}$ (Cor. 1, Prop. 43, no. 2)

we have, according to Banach theorem:

$\nu \lim s_n = \lambda \lim s_n$	$\mu\,(r_t) \lim s_n = \lambda \lim s_n$

if λ comes from a matrix; or

$\nu \lim s_n = \lambda\,(r'_t) \lim s_n$	$\mu\,(r_t) \lim s_n = \lambda\,(r'_t) \lim s_n$

if λ is continuous and $\{r'_t\}$ any arbitrary sequence of positive numbers with limit $+\infty$.

Therefore (Prop. 4, no. 2) it is surely

$\nu \lim s_n = \lambda \lim s_m$	$\mu\,(r_t) \lim s_n = \lambda \lim s_n$ whence (Prop. 4, no. 2). $\mu \lim s_n = \lambda \lim s_n$

Cor. *The conclusions of*

Prop. 2	Prop. 2′

and Theor. 1 *remain valid if the sequence*

$\{\alpha_t\, t^{(n+p)}\}$	$\{\alpha\,(r_t)\, r_t^{n+p}\}$

is complete with respect to **u** (Def. 5, no. 2, § 1).

Proof. Since then it is surely very weakly complete with respect to **u** (Prop. 5, no. 2, § 1).

7. *Identity of Euler and Borel limits with those of a stronger algorithm*

Def. 1. The classic convergence algorithm of

Euler	Borel

which we will design in short by

E_0	B_0

to be distinguished from the summation algorithm (Def. 1, no. 4), is, after (Def. 1, no. 6) a convergence algorithm of the $p = 0$ type of moments, with the generatrix

$a_t = 2^{-t}$ for $t = 0, 1, 2, \ldots$	$a(t) = e^{-t}$ for $0 \leq t < +\infty$

which has:

$\mu_n = -\sum_{t=0}^{\infty} t^{(n}, \Delta\alpha_t = \sum_{t=0}^{\infty} 2^{-t-1}\, t^{(n} = n!$	$\mu_n = -\int_0^{\infty} t^n\, a\, d\, a(t) = \int_0^{\infty} e^{-t}\, t^n\, dt = n!$

for $n = 0, 1, 2, \ldots$

Def. We will call a *convergence algorithm*

E_1,	B_1,

to the algorithm of the $p = 1$ type of moments with the same generatrix as

E_0	B_0

that is (Def. 1, no. 6) to the permanent linear algorithm with convergence factors

$\lambda_{t,n} = 2^{-t} \dfrac{t^{(n+1}}{(n+1)!}$ for $t = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$	$\lambda_n(t) = e^{-t} \dfrac{t^{n+1}}{(n+1)!}$ for $0 \leq t \leq +\infty$ or more general obtained by analytical continuation of the associated

The convenience of introducing such a convergence algorithm

$E_1 \neq E_0$	$B_1 \neq B_0$

is based on the following Prop. 1, which will be used in the proof of (Theor. 1, no. 8) below

Prop. 1. *The summation and convergence algorithms of*

Euler, $E = E_1$	*Borel*, $B = B_1$

(Def. 5, no. 1). (Def. 1, no. 4), (Def. 2, no. 7) are equivalent

and also, specially, their subordinates (Def. 3, no. 3, and Def. 5, no. 2)

$$B(r_t) = B_1(r_t)$$

through any arbitrary sequence $\{r_t\}$ *of positive numbers with limit* $+\infty$

Proof. [10, 338] [12, 212]. Writing

$$s_n = \sum_{\nu=0}^{n} a_\nu \text{ for } n = 0, 1, 2, \ldots,$$

let

$\{A_t\}$ and $\{A'_t\}$	$A(t)$ and $A_1(t)$

be the associated to the series $\sum_{n=0}^{\infty} a_n$ and to the sequence $\{s_n\}$ in the summation and convergence algotirlms

E and E_1	B and B_1

respectively (Def. 1, no. 3 and Def. 1, no. 6).

Since

$\Delta(2^{-t} A'_t) = 2^{-t-1} [(\Delta A'_t) - A'_t] = 2^{-t-1} A_t$ for $t = 0, 1, 2, \ldots$	$D[e^{-t} A_1(t)] = e^{-t} [A'_1(t) - A_1(t)] = e^{-t} A(t)$ for $0 \leq t < +\infty$

and furthermore

$\left[2^{-t} A_t\right]_{t=0} = 0,$	$\left[e^{-t} A_1(t)\right]_{t=0} = 0,$

we have

$\frac{1}{2} \sum_{\nu=0}^{t} 2^{-\nu} A_\nu = 2^{-t} A'_t$ for $t = 0, 1, 2, \ldots$ (1)	$\int_0^t e^{-\nu} A(\nu) d\nu = e^{-t} A_1(t)$ for $0 \leq t < +\infty$ (1′)

whence, for $t \to +0$, it follows:

$\sum_{n=0}^{\infty} a_n \in S_E \rightleftarrows \sum_{n=0}^{\infty} a_n \in S_{E_1}$	$\sum_{n=0}^{\infty} a_n \in S_E \rightleftarrows \sum_{n=0}^{\infty} a_n \in S_{B_1}$
$E \sum_{n=0}^{\infty} = E_1 \sum_{n=0}^{\infty} a_n$	$B \sum_{n=0}^{\infty} a_n = B_1 \sum_{n=0}^{\infty} a_n$

which are the two conditions (1) and (2) of (Def. 5, no. 1) for

E and E_1	B and B_1

Furthermore for every sequence $\{r_t\}$ of positive numbers

$$r_t \geq 0 \quad \text{for} \quad t = 0, 1, 2, \ldots$$

we have, specially, after (1′)

$$\int_0^{r_t} e^{-\nu} A(\nu)\, d\nu = e^{-r_t} A_1(r_t) \text{ for } t = 0, 1, 2, \ldots$$

then, if

$$\lim_{t=\infty} r_t = +\infty$$

it results for $t \to \infty$.

$$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_{B(r_t)} \rightleftarrows \sum_{n=0}^{\infty} a_n \in \mathbf{S}_{B_1(r_t)}$$

$$B(r_t) \sum_{n=0}^{\infty} a_n = B_1(r_t) \sum_{n=0}^{\infty} a_n$$

which are conditions (1) and (2) of the said (Def. 5, no. 1) for subordinate algorithms

$B(r_t)$ and $B_1(r_t)$

through any arbitrary sequence of positive numbers with limit $+\infty$

Lemma 1. *The conclusions of (Theor. 1, no. 6) and that of*

(Prop. 2, no. 6)	(Prop. 2′, no. 6)

are verified for the convergence algorithms

E_0 and E_1	B_0 and B_1 *and furthermore, every increasing sequence* $\{r_t\}$ *with limit* $+\infty$ *and terms strictly greater than one, i. e., with* $\lim_{t\to\infty} r_t = +\infty$ and $1 < r_t \leq r_{t+1}$ for $t = 0, 1, 2, \ldots$ *is regular (Def. 2′, no. 6) with respect to* B_0 *and* B_1.

Proof. By virtue of (Cor. Theor. 1, no. 6), it suffices to remember that the sequences

$\{2^{-t-1}\, t^{(n)}\}$ and $\{2^{-t-1}\, t^{(n+p)}\}$	$\{e^{-r_t}\, r_t^n\}$ and $\{e^{-r_t}\, r_t^{n+p}\}$

are complete with respect to u, as we saw in

(Ex. 1 and 1′, no. 2, § 1)	(Ex. 1″, no. 2, § 1) and furthermore, the second condition ((2′) no. 6) is evidently satisfied for every sequence $\{r_t\}$ whose terms are greater than 1, since the generatrix $e^{-t} > 0$, for $t > 0$.
Prop. 2. *The convergence algorithms* E_0 *and* E_1	Prop. 2′. *Every convergence algorithm subordinated to the convergence algorithm* B_0 *or* B_1 *through a regular sequence* $\{r_t\}$ *(Def. 2′, no. 6 and Lem. 1)*

is perfect.

Proof. According to Lemma 1, this proposition is included in

(Prop. 2, no. 6)	(Prop. 2′, no. 6)

Theor. 1. *If λ is a permanent convergence algorithm, with matrix or continuous (Def. 1, no. 2), stronger (Def. 3, no. 1) than*

the linear convergence algorithm of Euler, E_0 (Def. 1)	*the set or union (Def. 4, no. 1) of the subordinated algorithms (Def. 5, no. 2) to the linear convergence algorithm of Borel, B_0 (Def. 1) through regular sequences $\{r_t\}$ (Def. 2', no. 6),*

we have

$E_0 \lim s_n = \lambda \lim s_n$	$B_0 \lim s_n = \lambda \lim s_n$

for every sequence

$\{s_n\} \in \mathbf{s}_{E_0}$	$\{s_n\} \in \mathbf{s}_{B_0}$
E_0-convergent.	*B_0-convergent.*

Proof. It is included in (Theor. 1, no. 6) by virtue of Lemma 1.

8. *Identity of the Euler and Borel sums with those of a stronger algorithm.*

It is expressed by the following:

Theor. 1. *If a permanent, linear algorithm of summation μ, equivalent (Def. 5 no. 1) to a linear convergence algorithm λ, is stronger than*

the summation algorithm of Euler, E (Def. 1, no. 4)	*the set or union (Def. 4, no. 1) of the algorihtms $B(r_t)$ (Def., 5, no. 3) subordinated to the linear summation algorithm of Borel, B, through regular sequences $\{r_t\}$, (Def. 2', no. 6 and Lem. 1)*

we have

$E \sum_{n=0}^{\infty} a_n = \mu \sum_{n=0}^{\infty} a_n$	$B \sum_{n=0}^{\infty} a_n = \mu \sum_{n=0}^{\infty} a_n$

for every series

$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_E$	$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_B$
E-summable	*B-summable*

Proof. Being, by assumption

$E \subset \mu$;	$B(r_t) \subset \mu$ (Def. 4, no. 1)

and by virtue of (Prop. 1, no. 7):

$E_1 = E$	$B_1(r_t) = B(r_t)$

it results from (Prop. 4, no. 1):

$E_1 \subset \lambda$;	$B_1(r_t) \subset \lambda$;

and, since after (Lem. 1, no. 7), the (Theor. 1, no. 6) can be applied, we have

$E_1 \lim s_n = \lambda \lim s_n$	$B_1 \lim s_n = \lambda \lim s_n$

for every sequence

$\{s_n\} \in \mathbf{S}_{E_1}$	$\{s_n\} \in \mathbf{S}_{B_1}$

Whence, according to (Prop. 1, no. 7) and to (Def. 5, no. 1) of equivalence, it follows

$E \sum_{n=0}^{\infty} a_n = \mu \sum_{n=0}^{\infty} a_n$	$B \sum_{n=0}^{\infty} a_n = \mu \sum_{n=0}^{\infty} a_n$

for every sequence

$\sum_{n=0}^{\infty} a_n \in S_E$	$\sum_{n=0}^{\infty} a_n \in \mathbf{S}_B$

q. e. d.

Remark. The existence of permanent linear algorithm of summation not equivalent to any permanent linear algorithm of convergence was pointed out in [12, 136-137]. Therefore, it has been necessary in Theor. 1 the hypothesis of the equivalence $\mu = \lambda$.

9. *Analysis of the results of this paper and pending problems.*

The theorems in numbers 5, 7 and 8 explain the central rol plaid by the Euler's algorithms, and even more so by the Borel's one, in the summation of numerical series [7] [13].

The one in no. 5 carries a hypothesis of continuity for the sum σ with the specific topology (Def. 2, no. 4) of the algorithm

E,	B,

which is hard to prove and requires a special study for each algorithm.

But it has just been, thanks to this law of continuity with the specific topology

E	B

together with Abel's law (no. 1, § 2), that we have been able to give now an affirmative answer to the Kogbetliantz's question [8, 11]; which stated, as he did, without including the said continuity law f' (no. 1) had to be answered negatively [20, 1841], even postulating simultaneously, with Kogbetliantz, the two laws c'_1 and d' for suppression of the initial term and the product, in their most restricted forms, which requires

$$\sum_{n=1}^{\infty} a_n \in \mathbf{S}_\sigma \quad \text{or} \quad \sum_{n=0}^{\infty} (a_n * b_n) \in \mathbf{S}_\sigma$$

and are no more verified by so fundamental algorithms as that of Borel; and the same negative answer is valid by adding Abel's law, e'.

The said theorem of no. 5 completes for algorithms weaker than

E	B

that of no. 8 for algorithms stronger than

E	the set of subordinates of B through regular sequences

It remains, therefore, the problem for those algorithms

E	B

nor stronger than

E | the union of the subordinates to B through regular sequences

that is, for which

$\mathbf{S}_{\mu} \not\subset \mathbf{S}_{E}, \quad \mathbf{S}_{E} \not\subset \mathbf{S}_{\mu}$ | $\mathbf{S}_{\mu} \not\subset \mathbf{S}_{B}, \mathbf{S}_{\mu} \not\supset \bigcup (\mathbf{S}_{B(r_t)}, \{r_t\}$ regular)

Since, furthermore, this theorem of no. 8 assures only equality for the sum of the

E-summable series, | B-summable series,

it remains unsolved the coincidence of the sums for the series, which being not

E-summable, | B-summable,

may do it, nevertheless, with some algorithms stronger than u

E | B

This is the same problem that arised when passing from convergent series, in which coincide the sums by any permanent algorithm, to the summable ones, Ordinary convergence has been now substituted by the

E | B

summation, which is much wider, and even some other algorithms might be tried, according to the theorem no. 6.

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CORRIGENDA

THE UNIQUENESS PROBLEM IN THE THEORY OF NUMERICAL DIVERGENT SERIES AND FORMAL LAWS OF CALCULUS II

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Page 23 lines 14, 22, 24, and page 24 line 19, for «isomorphism» read «homomorphism».