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Estadística

Branching processes: Genealogy and Evolution

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Abstract

Branching process theory provides appropriate mathematical models to describe the probabilistic evolution of systems whose components (cells, particles, individuals in general) after a certain life period reproduce and die. The genealogy of this theory is presented, describing the basic results of the standard Bienaymé-Galton-Watson process and some of its generalizations, paying special attention to controlled branching processes. Several fields in which this theory has been applied successfully are indicated.

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1. Brief History of Branching Process Theory

Branching process theory is that part of mathematics which deals with the growth and decay of systems whose components reproduce following some stochastic laws. The term *Branching Process* appears to have been coined by A.N. Kolmogorov and N.A. Dmitriev in 1947 (see [35]) but the subject is much older and goes back to more than a century and a half ago. Initially it was motivated to explain the extinction phenomenon of certain family lines of European aristocracy.

It had traditionally been considered that the modern theory of branching processes was initiated in England by F. Galton with his classical formulation published in 1873 in the *Educational Times*:

> PROBLEM 4001: A large nation, of whom we will only concern ourselves with adult males, N in number, and who each bear separate surnames colonise a district. Their law of population is such that, in each generation, a_0 per cent of the adult males have no male children

who reach adult life; a_1 have one such male child; a_2 have two; and so on up to a_5 who have five. Find (1) what proportion of their surnames will have become extinct after r generations; and (2) how many instances there will be of the surname being held by m persons.

But, in 1977, C.C. Heyde and E. Seneta (see [28]) showed that, in France, notably L.F. Benoiston de Châteauneuf (see [6]) and I.J. Bienaymé (see [5]) had considered the nobility and family extinction problem before F. Galton publicized it. Indeed, they pointed out that I.J. Bienaymé was not only the first to formulate the mathematical problem, but indeed knew its solution already in 1845, although the original publication has not been found (see also [7]).

Coming back to Galton's problem, he persuaded his acquaintance H.W. Watson to work out a solution. Watson proposed a solution using the theory of generating functions and functional iteration, which they published together in 1874 (see [43]). However, they concluded erroneously that every family will die out, even when the population size, on average, increases from generation to generation. Even though, as was noted above, I.J. Bienaymé already knew a correct solution to the extinction problem, this remained unknown, and it took more that fifty years to rectify the solution provided by H.W. Watson. The first complete and correct determination of the extinction probability following Galton-Watson's line was given by J.F. Steffensen (see [39] and [40]). He established that if the mean number of children is less than or equal to one, then Galton and Watson were right, but if it is greater than one, the extinction probability of the family name is less than unity.

In parallel with Steffensen's studies, branching processes appear in scientific literature in the 1920s and 1930s with the works of R.A. Fisher (see [13]) and J.B.S. Haldane (see [25]) who applied them to study the problem of the dispersion and extinction of a mutated gene in a population. From then on, the original model introduced by Bienaymé, Galton, and Watson, and its generalizations have been treated extensively for their mathematical interest and as theoretical approaches to solving problems in applied fields such as Biology (gene amplification, clonal resistance theory of cancer cells, polymerase chain reactions,...), Epidemiology, Genetics, and Cell Kinetics (the evolution of infectious diseases, sex-linked genes, stem cells,...), Computer Algorithms and Economics, and, of course, Population Dynamics, to mention only some of the more important applications.

For a detailed history of branching processes see [30] and [31] or the classical references [1], [2], [3], [26], [27], [29], and [37]. Also P. Jagers' notes (see http://www.math.chalmers.se/~jagers/Branching%20History.pdf) of his lecture given at the 2009 Oberwolfach Symposium on "Random Trees" are very amusing. Moreover, for those wanting to acquaint themselves with contemporary branching process theory and its applications, see the recent monographs

[4], [22], [23], [33], and [36]. All this has made branching process theory one of the most dynamic fields in the general theory of stochastic processes.

Having given an overview of the genealogy of branching processes, our aim in the rest of this communication is to introduce the mathematical formulation of the standard Bienaymé–Galton–Watson branching process (BGWP), to describe how it has been modified to tackle more complex real problems, and to indicate some of its applications. Briefly, Section 2 is devoted to defining the BGWP, its main properties, and some of its generalizations. In Section 3, we focus on one of these modifications that has aroused our especial interest: controlled branching processes. We state the main results without proofs, and invite the reader to look more deeply into them by consulting the references provided.

2. Bienaymé-Galton-Watson Branching Processes and their generalizations

With the nomenclature of population dynamics, a BGWP is a discrete-time stochastic process that describes the evolution of a population in which each individual independently of the others gives rise to a random number of offspring (in accordance with a common reproduction law), and then dies or is not considered in the following counts. We shall give its formal definition and establish some interesting properties. Let $\{X_{nj} : n = 0, 1, \ldots; j = 0, 1, \ldots\}$ be non-negative integer valued independent and identically distributed (i.i.d.) random variables with probability distribution $\{p_k\}_{k \geq 0}$, i.e. $P(X_{01} = k) = p_k$, $k \geq 0$. The BGWP is a stochastic process, $\{Z_n\}_{n \geq 0}$, defined recursively as follows:

$$Z_0 = N \in \mathbb{N}, \quad Z_{n+1} = \sum_{j=1}^{Z_n} X_{nj}, \ n \geq 1, \tag{2.1}$$

where $\sum_{j=1}^{0}$ is defined to be 0. Thus, X_{nj} represents the number of offspring produced by the j-th individual in the n-th generation, and Z_n represents the number of individuals in the n-th generation. We refer to $\{p_k\}_{k \geq 0}$ as the offspring distribution or law, with p_k being interpreted as the probability that an individual has k offspring. The expected value and the variance of the offspring distribution, denoted by m and σ^2, are called the offspring mean and variance, respectively.

It is obvious that if the size of the n-th generation is known then the probability law governing later generations does not depend on the sizes of the generations before the n-th. Hence the BGWP is a Markov chain with 0 an absorbing state. Moreover, since each individual reproduces in accordance with the same offspring distribution, the transition probabilities are stationary.

A fundamental feature of a BGWP is the well-known additivity property. We have defined $Z_0 = N \in \mathbb{N}$, but in the following we shall always assume that

$Z_0 = 1$, because the BGWP starting with N individuals behaves as the sum of N independent BGWP processes all starting with one individual (see [29]). Moreover, to avoid trivialities, one usually assumes that $p_0 < 1$ and $p_0 + p_1 < 1$.

By the extinction of the process is meant the event $Q = \{Z_n \to 0\}$, i.e., the set $\{Z_n = 0 \text{ eventually}\}$, which is obviously the same as $Z_n = 0$ for some n. The problem originally set out by F. Galton was to find the probability of the extinction of a family, i.e., to determine $P(Q)$. We shall denote this probability by q. Notice that $q = \lim_{n\to\infty} P(Z_n = 0) = \lim_{n\to\infty} F_n(0)$, with $F_n(s) = E[s^{Z_n}]$, $0 \leq s \leq 1$, being the probability generating function of Z_n. Moreover it can be proved that $q = \lim_{n\to\infty} F_n(s)$, $0 \leq s < 1$. The solution to the extinction problem is established in terms of m and of the probability generating function of the offspring law, denoted by $f(s)$. In particular, the extinction probability q is stated to be the smallest non-negative root of the equation

$$f(s) = s. \tag{2.2}$$

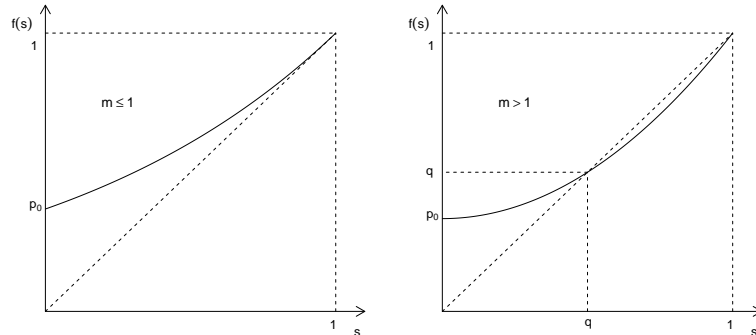If $m \leq 1$ then $q = 1$, and if $m > 1$ then $0 \leq q < 1$ (see Figure 1).

Figure 1: Probability generating function and extinction probability.

The sequence $\{Z_n\}_{n\geq 0}$ does not remain positive and bound. It either goes to 0 or goes to ∞. This property is known as the extinction-explosion duality, i.e.,

$$P(Z_n \to 0) + P(Z_n \to \infty) = 1. \tag{2.3}$$

As a consequence of the extinction result, BGWPs with $m < 1$, $m = 1$, and $m > 1$ are called subcritical, critical, and supercritical, respectively. Figure 2 shows their different behaviour with respect to extinction.

Another problem of interest is to study the asymptotic distribution of Z_n when n is large.

In the subcritical case, the process dies out with probability one. To describe its asymptotic behaviour, one introduces the device of imposing the condition
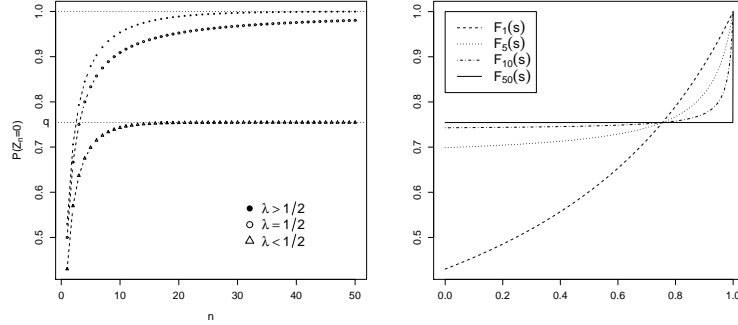
Figure 2: Left: Extinction probabilities for BGWPs with geometric offspring distribution of parameter λ, i.e., $p_k = (1 - \lambda)^k \lambda$, $k \geq 0$. In this case $m = (1 - \lambda)\lambda^{-1}$. Thus, according to whether $\lambda > 2^{-1}$, $= 2^{-1}$, or $< 2^{-1}$, the process is subcritical, critical, or supercritical, respectively. Right: Evolution of the probability generating functions of Z_n in the supercritical case.

that extinction has still not happened, and one obtains that, for $k \in \mathbb{N}$,

$$\lim_{n \to \infty} P(Z_n = k \mid Z_n > 0) = b_k, \tag{2.4}$$

with $\sum_{k=1}^{\infty} b_k = 1$ and $\sum_{k=1}^{\infty} k b_k < \infty$ iff $\sum_{k=1}^{\infty} k p_k \log k < \infty$. This result was given by A.M. Yaglom (see [44]).

The unsteadiness of the critical case is clearly shown due to the fact that $E[Z_n] = 1$ for all $n \in \mathbb{N}$, $Var[Z_n] \to \infty$, as $n \to \infty$, and that $Z_n \to 0$ with probability one. In this case the limit probabilities b_k in (2.4) are null, so that a further normalization is need to make the conditional process converge to a non-degenerate limit. Thus, A.N. Kolmogorov (see [34]) and A.M. Yaglom (see [44]) proved that if $\sigma^2 < \infty$ then, for all real numbers z,

$$\lim_{n \to \infty} P\left(n^{-1} Z_n \leq z \mid Z_n > 0\right) = \Gamma_{a,b}(z),$$

with $\Gamma_{a,b}$ denoting the gamma distribution function with parameters $a = 1$ and $b = 2^{-1}\sigma^2$. In other words, if the process has not become extinct and n is large, then the process has a linear growth and the distribution of $n^{-1} Z_n$ is almost exponential. Moreover, the following estimate of the rate of convergence to zero is proved:

$$P(Z_n > 0) \sim 2\sigma^2 n^{-1}, \text{ as } n \to \infty. \tag{2.5}$$

Finally, using martingale theory one can establish the behaviour of Z_n in the supercritical case. In particular, attention is focused on the process $\{m^{-n} Z_n\}_{n \geq 0}$. It is verified that, for $0 < m < \infty$, $m^{-n} Z_n \to W$ almost surely as $n \to \infty$. One

knows that $P(W = 0) = 1$ for $m \leq 1$. However, K. Kesten and B.P. Stigum (see [32]) proved that for $m > 1$

$$P(W > 0) > 0 \quad \text{iff} \quad \sum_{k=1}^{\infty} k p_k \log k < \infty$$

and

$$P(W > 0) = P(Z_n \to \infty) = 1 - q.$$

This result implies that on the non-extinction set, $Z_n \sim m^n W$, and hence one should expect the population eventually to grow at a geometric rate.

The proofs of the extinction and limiting results can be found, for instance, in [1], [3], and [29].

Finally, a further topic of a great interest is the inferential theory arising from a BGWP. In this sense, the behaviour of estimators for the offspring mean and variance and the extinction probability has been described and investigated. A good exposition of the main results on this topic is given in [24].

Despite the simplicity of the model, there have been recent applications of the BGWP as can been found for instance in [12], which considers the problem of modeling outbreaks of infectious disease, and in [9], which treats the problem of populating an environment.

The simple reproduction scheme considered in the BGWP can be generalized, giving rise to other families of branching processes that could allow complex practical situations to be reasonably modeled when the BGWP can not provide an acceptable description. Thus, in the second half of the XX century, new branching processes were introduced in both discrete and continuous time. We shall present some of them. One can go more deeply into their study by reading the monographs [1], [2], [26], [27], [29], and [37].

One straightforward generalization leads to multitype branching processes. These consider K different types of individuals, and the process is determined by a collection of $f_i(s_1, \ldots, s_K)$, $i = 1, 2 \ldots, K$, generating functions for which the coefficient of $s_1^{i_1} s_2^{i_2} \cdots s_K^{i_K}$ in f_i is the probability of an individual of type i giving rise to i_j individuals of j-th type, $j = 1, \ldots, K$. The role of the offspring mean, m, in the BGWP is now played by a $K \times K$ matrix M of which the element (i, j) is the expected value of the number of individuals of type j produced by an individual of type i.

In the simple model, we have assumed that individuals reproduce independently of each other, and with the same offspring distribution. In the real world, these properties can be violated in several ways. One can consider that the offspring number depends on population size or that it can be vary over generations because of factors such as food supply. Thus there appear population size dependent branching processes, branching processes in varying environments, and branching processes in random environments. In brief, the future evolution

of these kinds of branching processes is determined by a sequence of offspring means.

One can also consider the migration phenomenon instead of isolated or closed populations that evolve from a given number of ancestors. This introduces branching processes with immigration and controlled branching processes. Due to our own special interest in the latter, these will be presented in greater detail in the following section.

Up to now we have ignored the fact that in species with sexual reproduction changes in the population sizes depend on the formation of couples. In many populations, mating is an important factor that can not be neglected. Bisexual branching processes take this into account explicitly. In general, these processes start with N couples. Each couple has random numbers of female and male offspring which form new couples in accordance with a deterministic or stochastic function, and so on. Specific reviews of bisexual branching processes can be found in Chapter 2 in [23] and Chapter 20 in [22].

All the previous models are discrete-time branching processes. Time structure was added to the simple reproductive scheme of a BGWP in Bellman–Harris branching processes, also called age-dependent branching processes. These describe populations in which individuals may have variable life spans, and split into a random offspring number at death, independently of age. The idea that the offspring law could also depend on age was actually introduced in Sevastyanov branching processes. Finally, the general way to describe reproduction is to consider that offspring can be born at several ages during a potential progenitor's life, and each birth event can result in single or multiple offspring. This model is called a Crump–Mode–Jagers branching process.

There are many applications of the above processes to real problems. To cite some of the more recent, we would direct the reader to Chapters 12–18 in [22] which present very interesting applications of some generalizations of the BGWP in Cell Kinetics, Genetics, and Epidemiology. In particular, they consider new ideas for branching process theory that arise in modeling leukæmia cell kinetics, (in vitro) progenitor cell populations, the amplification, mutation, and selection forces of Alu elements, the evolution of the number of carriers of the two alleles of a gene linked to the Y chromosome, the spread of an SIR (susceptible–infective–removed) epidemic among a closed, homogeneously mixing population consisting initially of certain numbers of infective and susceptible individuals, the propagation of *Bovine Spongiform Encephalopathy* at the scale of a very large population, outbreaks of infectious diseases with an incubation period, and the transmission dynamics of the macroparasite *Echinococcus granulosus*.

3. Controlled branching processes

The model considered in this section is the controlled branching process. This is a generalization of the classical BGWP, and is used to describe the evolution of populations which require control of the population size at each generation. This consists of determining the number of individuals with reproductive capacity at each generation mathematically through a control process. In practice, this branching model could describe reasonably the probabilistic evolution of populations in which, for various reasons of an environmental, social, or other nature, there is a mechanism that establishes the number of progenitors which take part in each generation. For example, in an ecological context, one can think of an invasive animal species that is widely recognized as a threat to native ecosystems, but there is disagreement about plans to eradicate it, i.e., while the presence of the species is appreciated by a part of society, if its numbers are left uncontrolled it is known to be very harmful to native ecosystems. In such a case, it is better to control the population to keep it, for example, between admissible limits even though this might mean periods when animals have to be culled. Two examples of recent discussions about this are given in [11] and [42].

We shall now present the formal definition of controlled branching processes and their main results by comparing them with the BGWP. B.A. Sevastyanov and A.M. Zubkov in 1974 (see [38]) introduced a process with the novelty of incorporating into the probability model a deterministic control function that fixes the number of progenitors generation by generation. Subsequently, N.M. Yanev (see [45]) generalized this model by considering a random control function. We shall give the precise definition of Yanev's model. A controlled branching process (CBP) with a random control function is a stochastic process, $\{Z_n\}_{n\geq 0}$, defined recursively as follows:

$$Z_0 = N \in \mathbb{N}, \quad Z_{n+1} = \sum_{j=1}^{\phi_n(Z_n)} X_{nj} \quad n \geq 0, \tag{3.1}$$

where, as in the BGWP, $\{X_{nj} : n = 0, 1, \ldots; j = 0, 1, \ldots\}$ are non-negative integer valued i.i.d. random variables (we keep the notation $p_k = P(X_{nj} = k)$), and, for each $n = 0, 1, \ldots$, $\{\phi_n(k)\}_{k\geq 0}$ are independent stochastic processes with equal one-dimensional probability distributions. As before, the random variable Z_n represents the total number of individuals in generation n, starting with $Z_0 = N > 0$ progenitors. Each individual, independent of all others and all with identical probability distributions, gives rise to new individuals. The random variable X_{nj} is the number of offspring originated by the j-th individual of generation n. The novel point is that if in a certain generation n there are k individuals, i.e., $Z_n = k$, then, through the random variable $\phi_n(k)$, identically distributed for each n, there is produced a control in the process fixing the num-

ber of progenitors which generate Z_{n+1}. Thus the variable $\phi_n(k)$ determines the migration process in a generation of size k: for those values of the variable $\phi_n(k)$ such that $\phi_n(k) < k$, $k - \phi_n(k)$ individuals are removed from the population, and therefore do not participate in the future evolution of the process; if $\phi_n(k) > k$, $\phi_n(k) - k$ new individuals (immigrants) of the same type are added to the population participating as progenitors under the same conditions as the others. No control is applied to the population when $\phi_n(k) = k$. The model introduced by B.A. Sevastyanov and A.M. Zubkov is the particular case of setting $\phi_n(k) = \varphi(k)$, with φ being a function that is non-negative and integer-valued for integer-valued arguments.

Focusing on definition (3.1), one observes that the CBP with a random control function is also a homogeneous Markov chain. However the additivity property is not always satisfied, hence the importance of fixing the initial number of progenitors. Now the state 0 is absorbing iff $\phi_n(0) = 0$ almost surely. In this case, the extinction-explosion duality (2.3) is verified if at least one of the following conditions holds: $p_0 > 0$, or $P(\phi_0(k) = 0) > 0$, $k = 1, 2, \ldots$. Thus the extinction problem has sense under the assumption of $\phi_n(0) = 0$ almost surely.

The presence of the random control function complicates the solution of the extinction problem. This topic was initially addressed by N.M. Yanev (see [45]) and F.T. Bruss (see [8]) by considering $\phi_n(k) = \alpha_n k(1 + o(1))$ almost surely as $n \to \infty$, where $\{\alpha_n\}_{n \geq 0}$ is a sequence of non-negative i.i.d. random variables.

M. González, M. Molina, and I. del Puerto (see [16]-[20]) studied the extinction problem of these processes from a more general outlook than that considered by N.M. Yanev and F.T. Bruss. In particular, let $m = E[X_{01}]$, $\sigma^2 = Var[X_{01}]$, $\varepsilon(k) = E[\phi_0(k)]$, and $\nu^2(k) = Var[\phi_0(k)]$, $k = 0, 1, \ldots$, and define

$$\tau(k) = E[Z_{n+1} Z_n^{-1} \mid Z_n = k] = mk^{-1} \varepsilon(k), \ k = 1, 2, \ldots.$$

Intuitively $\tau(k)$ is interpreted as the expected growth rate per individual when, in a certain generation, there are k individuals. In order to obtain conditions for the almost sure extinction and for the existence of a positive probability of non-extinction, different possible behaviours of the sequence $\{\tau(k)\}_{k \geq 1}$ with respect to 1 were considered. Extending the classification provided by m in a BGWP, in a broad sense the cases $\limsup_{k \to \infty} \tau(k) < 1$, $\liminf_{k \to \infty} \tau(k) \leq 1 \leq \limsup_{k \to \infty} \tau(k)$, and $\liminf_{k \to \infty} \tau(k) > 1$ are referred to, respectively, as subcritical, critical, and supercritical situations for a CBP with a random control function.

In the subcritical case, the process dies out with probability one independently of the initial number of progenitors (see [16]).

In general, and unlike the situations of the critical and supercritical BGWPs, neither the extinction with probability one in the former case nor the non-extinction with positive probability in the latter case are always guaran-

teed. First, we shall describe the supercritical case, and then the critical one.

In the supercritical case, if $\{k^{-1}\varepsilon(k)\}_{k\geq1}$ and $\{k^{-1}\nu^2(k)\}_{k\geq1}$ are bounded sequences, there exists $N_0 \in \mathbb{N}$ such that if the number of initial progenitors is greater than N_0 then there is non-extinction of the process with positive probability (see [16]). If the condition on the control variances, $\nu^2(k)$, is not satisfied then the extinction of the process could occur with probability one.

The behaviour of the critical case in a CBP with random control function is much richer than in a BGWP. Indeed, this case is vast. We have as yet only considered the situation in which the limit of $\{\tau(k)\}_{k\geq1}$ exists and is equal to unity. We have distinguished two situations based on the speed of convergence of $\{\tau(k)\}_{k\geq1}$. On the one hand, if the convergence of $\{\tau(k)\}_{k\geq1}$ to 1 is very fast (with respect to the conditional variance), the whole process behaves as a true critical process in the sense of a BGWP, and almost sure extinction follows. On the other hand, if the convergence of $\{\tau(k)\}_{k\geq1}$ is very slow, the process develops as a simple supercritical process, having a positive probability of non-extinction. Let us introduce the notation $\ell_2(k)$ for the conditional variance of the process, i.e., $\ell_2(k) = Var[Z_{n+1} \mid Z_n = k]$. Under some technical conditions, which the reader can find in [19] and are not repeated here for the sake of readability, we proved the following results on the extinction problem in the critical case. It is verified that if

$$\limsup_{k\to\infty} \frac{2(\tau(k)-1)}{\ell_2(k)k^{-2}} < 1,$$

then the process dies out with probability one independently of the initial number of progenitors. On the contrary, if

$$\liminf_{k\to\infty} \frac{2(\tau(k)-1)}{\ell_2(k)k^{-2}} > 1,$$

then the non-extinction of the process with positive probability occurs independently of the initial number of progenitors.

Figure 3 shows the behaviour of the extinction probability of a critical CBP under non-extinction conditions. An interesting open problem in the non-extinction cases is to determine exactly the extinction probability, q. As in BGWP, it can be proved that $q = \lim_{n\to\infty} F_n(s)$, $0 \leq s < 1$ (see Figure 3), but unfortunately there exists no equation analogous to (2.2) that provides q.

The above results provide the answer to the extinction problem for the most typical cases in this class of controlled models. They unfortunately do not cover certain situations, for instance, when

$$\lim_{k\to\infty} \frac{2(\tau(k)-1)}{\ell_2(k)k^{-2}} = 1. \tag{3.2}$$

Research in this direction proceeded by assuming that $\phi_0(k)$, $k = 1, 2, \ldots$, have
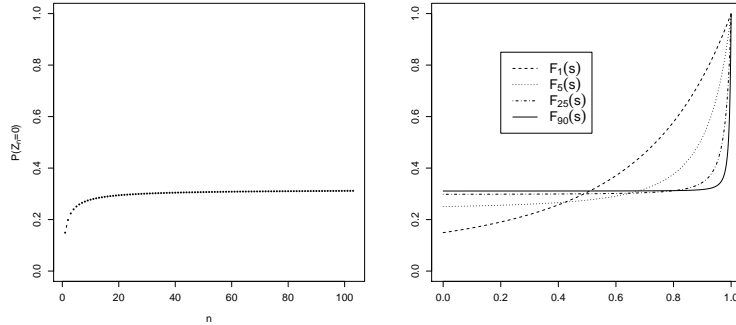
Figure 3: Left: Approximation to the extinction probability for a critical CBP whose offspring law is a Poisson distribution with parameter one, and control variables, $\phi_n(k)$, $k > 0$, are Poisson distributions with parameters $k + 2.01$ and $\phi_n(0) = 0$. Right: Evolution of the probability generating functions of Z_n.

infinitely divisible probability distributions, and setting

$$\tau(k) = 1 + k^{-1}c, \ c > 0, \ k = 1, 2, \ldots \qquad \text{and} \qquad \ell_2(k) = \nu k + O(1) \quad (3.3)$$

(consequently $\lim_{k \to \infty} 2(\tau(k) - 1)(\ell_2(k)k^{-2})^{-1} = 2c\nu^{-1}$). In these cases, one has that if $2c\nu^{-1} \leq 1$, then the process dies out. Moreover, if $2c\nu^{-1} < 1$ then

$$P(Z_{n-1} > 0) \sim k_1 n^{2c\nu^{-1}-1}, \ n \to \infty,$$

whereas $2c\nu^{-1} = 1$ implies

$$P(Z_{n-1} > 0) \sim k_2 (\log n)^{-1}, \ n \to \infty$$

for some positive constants k_i, $i = 1, 2$. It is therefore proved that for the critical CBP $P(Z_n > 0)$ decays to zero more slowly than for the critical BGWP – see (2.5). This is due to the immigration component – note that, by the assumption $\tau(k) = 1 + k^{-1}c, \ c > 0, \ k = 1, 2, \ldots$, there exists an expected immigration of individuals in each generation.

Next, we shall briefly describe the main results on the asymptotic behaviour of CBP.

In the subcritical case, under the assumption that zero is not an absorbing state, i.e., $P(\phi_n(0) > 0) > 0$, and $p_0 > 0$ and $p_0 + p_1 < 1$, it is verified that $\{Z_n\}_{n \geq 0}$ converges in distribution to a positive, finite, and non-degenerate random variable Z as $n \to \infty$. See details in [18].

With respect to the supercritical case, as in the BGWP, conditions to guarantee geometric growth on the whole set for which a CBP goes to infinity are

provided in [17], [20], and [21]. In particular, an in-depth study is made of the almost sure, in L^1 and L^2, convergence to a non-degenerate limit of the sequence $\{Z_n(\tau m)^{-n}\}_{n\geq 0}$, with $\tau = \lim_{k\to\infty} \varepsilon(k)k^{-1}$ and $\tau m > 1$.

The most surprising case is again the critical one. Different kinds of limiting behaviour are obtained for $\{Z_n\}_{n\geq 0}$ suitably normed. Critical CBPs which do not become extinct with a positive probability are considered. It is also assumed that $\tau(k) = 1 + ck^{\alpha-1} + o(k^{\alpha-1})$, where $\alpha < 1$, $c > 0$ and $\tau(k) > 1$, and $\ell_2(k) = \nu k^\beta + o(k^\beta)$, $\beta \leq \alpha + 1$, $\nu > 0$. It is necessary to introduce the function $g(k) = k(\tau(k) - 1)$, $k \geq 1$, and to extend it to a twice continuous differentiable function on $\mathbb{R}$. Let us denote by $\{a_n\}_{n\geq 0}$ the solution of the deterministic recursive equation

$$a_0 = 1, \quad a_{n+1} = a_n + g(a_n), \quad n = 0, 1, \ldots,$$

which plays an important role in the asymptotic behaviour of the process as will be seen below. It is verified that $a_n \sim (c(1-\alpha)n)^{(1-\alpha)^{-1}}$, as $n \to \infty$.
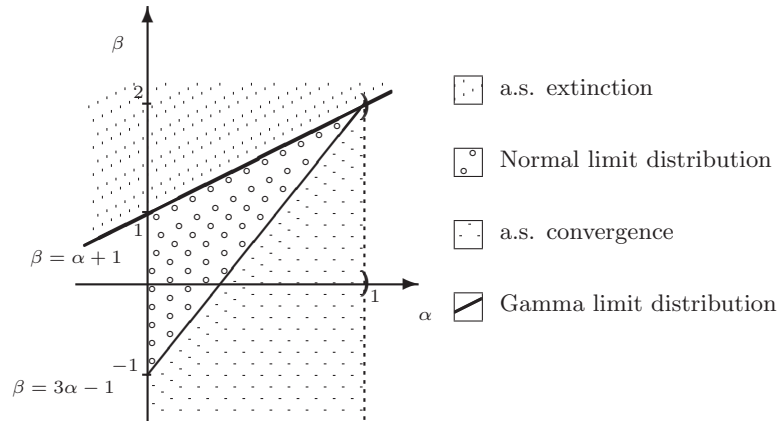

Figure 4: Asymptotic behaviour in the critical case.

Again, avoiding the more technical hypotheses – see [19] – one has:
If $\beta = \alpha + 1$ and $2c\nu^{-1} > 1$, for all real numbers z

$$\lim_{n\to\infty} P\left(n^{-1}Z_n^{1-\alpha} \leq z \mid Z_n > 0\right) = \Gamma_{a,b}(z),$$

with parameters $a = (\nu - \nu\alpha)^{-1}(2c - \nu\alpha)$ and $b = 2^{-1}\nu(1-\alpha)^2$.

If $0 < \alpha < 1$ and $\beta < \alpha + 1$ then one has that

- For $\beta < 3\alpha - 1$, on $\{Z_r \to \infty\}$, as $n \to \infty$

  $a_n^{-1} Z_n \to 1$ almost surely, and $g(a_n)^{-1}(Z_n - a_n)$ converges almost surely.

- For $\beta \geq 3\alpha - 1$, on $\{Z_r \to \infty\}$, $a_n^{-1} Z_n$ converges in probability to unity

as $n \to \infty$, and for all real numbers z,

$$\lim_{n \to \infty} P\left(\Delta_n^{-1/2} \frac{Z_n - a_n}{g(a_n)} \leq z \mid Z_n > 0\right) = \phi^*(z),$$

with ϕ^* being the standard normal distribution function, and

$$\Delta_n = \begin{cases} \nu c^{-3}(1-\alpha)^{-1} \log n & \text{if } \beta = 3\alpha - 1. \\[2ex] \nu(\beta - 3\alpha + 1)^{-1} c^{\frac{\beta-2}{1-\alpha}}((1-\alpha)n)^{\frac{\beta-3\alpha+1}{1-\alpha}} & \text{if } \beta > 3\alpha - 1. \end{cases}$$

Figure 4 shows, according to the α and β values, a simplified scheme corresponding to the different kinds of limiting behaviour obtained for $\{Z_n\}_{n \geq 0}$.

Finally, in the case in which one assumes that $\phi_0(k)$, $k = 1, 2, \ldots$, have infinitely divisible probability distributions and (3.3), if $2c\nu^{-1} \leq 1$ then

$$\lim_{n \to \infty} P(n^{-1} Z_n \leq z \mid Z_n > 0) = \Gamma_{a,b}(z),$$

with $a = 1$ and $b = 2^{-1}\nu$. Note that in this situation the parameter a does not depend on c or ν, unlike the analogous result corresponding to the non-almost-sure-extinction case. This result is similar to that of Kolmogorov and Yaglom concerning the limiting exponential distribution for the critical BGWP. Both models have the same exponential limiting distribution on their non-extinction sets, notwithstanding the different decay rates of their non-extinction probabilities.

With respect to the estimation theory for the CBP defined by (3.1), several approximations have been considered. By setting a fixed control function $\phi_n(k) = \varphi(k)$, M. González, R. Martínez and I. del Puerto (see [14] and [15]) considered the supercritical case, and made a non-parametric estimation of the offspring distribution and its parameters. J.P. Dion and E. Essebbar (see [10]) considered a particular CBP with a random control function $\phi_n(Z_n) = \alpha_n\varphi(Z_n)$, where $\{\alpha_n\}_{n \geq 0}$ is a sequence of i.i.d. random variables with positive integer values, and $E[\alpha_0] = \alpha$. They provided an estimator of $\theta = m\alpha$, and studied its behaviour in the supercritical case. In an attempt to solve the problem of providing estimators of the parameters of CBP which do not require any prior knowledge of the growth behaviour of the process, T.N. Sriram et al. (see [41]) proposed an estimation theory based on a conditional weighted least squares approach. They studied the asymptotic behaviour of the proposed estimator in the subcritical, critical, and supercritical cases. This theory remains unclosed. It is a very interesting open problem to continue developing it, dealing with the problems from bootstrap or sequential perspectives or those based on Markov

chain Monte-Carlo (MCMC) methods.

We shall finish by briefly noting some other open problems which become CBPs in an interesting field to investigate. With respect to the extinction problem, more effort is required to clarifying the critical case. Situations in which the existence of the limit of the expected growth rates is not guaranteed need to be dealt with. Other more complex models have been derived from (3.1) in a similar way to what was done for the BGWP. For instance, there appear multitype CBPs, and CBPs with population-size-dependent reproduction or in random environments. There has as yet been little work tackling the study of these new controlled models. Their development will open up an extensive area for research.

Acknowledgements

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