2. ARTÍCULOS DE INVESTIGACIÓN OPERATIVA

ON SOLUTION CONCEPTS FOR MULTI-CHOICE COOPERATIVE GAMES

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Abstract

This paper deals with the model of multi-choice games, a natural extension of the traditional model of cooperative games with transferable utility. Cooperative multi-choice game theory is a booming research topic with many recent developments on which this paper intends to offer a brief overview.

Keywords: Cooperative games, multi-choice games, solution concepts.

1. Solution concepts for arbitrary multi-choice games

Multi-choice cooperative games are introduced in Hsiao and Raghavan [10], [11] to allow cooperating players to be active at more than one level, under the assumption that the same number of participation levels is available for all players. The model of multi-choice games is considered in a more general setting in Nouweland [15] and Nouweland, Potters, Tijs, and Zarzuelo [16], where the number of participation levels for different players may be different. Building blocks for multi-choice games are the so-called multi-choice coalitions which are players' participation profiles available when a maximal participation profile is known. A real-valued characteristic function on the set of multi-choice coalitions quantifies the benefit of cooperation according to any participation profile; it is assumed that overall abstention from cooperation (i.e. cooperation at level 0) generates worth 0. A multi-choice game is a triplet $\langle N, m, v\rangle$ specifying the set $N = \{1, ..., n\}$ of players, their maximal participation profile $m = (m_1, ..., m_n)$ with $m_i \in \mathbb{Z}_+$ for each $i \in N$, and the characteristic function $v : \mathcal{M}^N \to \mathbb{R}$, $v(0) = 0$, where $\mathcal{M}^N$ stands for the set of multi-choice coalitions, that is the set of participation profiles s smaller than or equal to m.

Here is an example of a multi-choice game $\langle N, m, v\rangle$, where $N=\{1, 2\}$, $m=(2, 1)$, $v((0, 0))=0$, $v((1, 0))=5$, $v((2, 0))=6$, $v((0, 1))=3$, $v((1, 1))=9$, $v((2, 1)) = 13$.

Often, a multi-choice game is identified with its characteristic function. Let us denote by $MC^{N,m}$ the set of multi-choice games with a fixed finite set of players N and maximal participation profile m. Examples of application of the multi-choice game model to various situations can be found in Nouweland [15], Calvo and Santos [6], Peters and Zank [17]. Multi-choice cooperative games have been a useful tool for modeling interaction of players in economic and operations research situations in which they may have different options for cooperation, varying from non-cooperation (participation level 0) to a maximal participation level which is greater than or equal to 1. In particular, multi-choice games can be seen as an appropriate analytical tool for modeling cost allocation situations in which commodities are indivisible goods that are only available at certain finite number of levels. Clearly, when all the players can only abstain from cooperation or be active at level 1 we obtain the traditional model of cooperative games. Consequently, solution concepts on $MC^{N,m}$ appear as natural extensions of well-known solution concepts on the set G^N of traditional cooperative games with player set N. A basic notion for defining various solutions for multi-choice games is that of (level) payoff vector. A (level) payoff vector is a function $x : M \to \mathbb{R}$, where $M = \{(i, j) \mid i \in N, j \in M_i^+\}$ with $M_i^+ = \{1, ..., m_i\}$, which specifies for each player $i \in N$ and each of his levels $j \in M_i^+$

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the payoff to player i corresponding to a change of its activity level from $j-1$ to j. By convention, we define $x_{i0}=0$ for all $i \in N$. An example of a (level) payoff vector for the two-person multi-choice game previously presented is $(5,1,7)$, where $x_{11}=5$, $x_{12}=1$, $x_{21}=7$ ($x_{10}=x_{20}=0$). For each $s \in \mathcal{M}^N$, the payoff of s according to x is $X(s)=\sum_{i\in N}\sum_{j=1}^{s_i} x_{ij}$, and the payoff of player i for acting at level s_i is $X_{is_i}=\sum_{j=1}^{s_i} x_{ij}$. Appealing properties for (level) payoff vectors are: *efficiency*, i.e. $X(m)=v(m)$; *level-increase rationality*, i.e. for each $i \in N$ and each $j \in M_i^+$, x_{ij} is at least the increase in payoff that player i can obtain working alone when he changes his activity level from $j-1$ to j; *coalitional stability*, i.e. $X(s) \geq v(s)$ for each $s \in \mathcal{M}^N$. Let $v \in MC^{N,m}$.

The *imputation set* $I(v)$ of v consists of all efficient and level-increase rational (level) payoff vectors, that is

$$\begin{aligned}I(v)=\{x: M \to \mathbb{R} \mid X(m)=v(m);\\ x_{ij} \geq v(je^i)-v((j-1)e^i),\ i \in N,\ j \in M_i^+\},\end{aligned}$$

where e^i is the unitary vector with $e_k^i=0$ for all $k \neq i$ and $e_i^i=1$.

The *core* $C(v)$ of v consists of those imputations which are coalitional stable, that is

$$C(v)=\{x \in I(v) \mid X(s) \geq v(s) \text{ for all } s \in \mathcal{M}^N\}.$$

The *precore* $\mathcal{PC}(v)$ of v consists of all efficient and coalitional stable (level) payoff vectors, that is

$$\begin{aligned}\mathcal{PC}(v)=\{x: M \to \mathbb{R} \mid X(m)=v(m);\\ X(s) \geq v(s) \text{ for all } s \in \mathcal{M}^N\}.\end{aligned}$$

The *minimal core* $C_{\min}(v)$ of v consists of those core elements x for which do not exist other elements $y \in C(v)$ which are weakly smaller than x in the sense that $Y(s) \leq X(s)$ holds for each $s \in \mathcal{M}^N$, that is

$$\begin{aligned}C_{\min}(v)=\{x \in C(v) \mid \nexists y \in C(v) \text{ s.t. } y \neq x\\ \text{and } y \text{ is weakly smaller than } x\}.\end{aligned}$$

By considering a domination relation on $I(v)$ based on players' levels of activity, the notions of *dominance core* and *stable set* are introduced in Nouweland et al. [16] as natural extensions of their traditional counterparts.

Let $s \in \mathcal{M}^N \setminus \{0\}$ and $x, y \in I(v)$. We say that the imputation y dominates the imputation x via coalition s, denoted by $y\, dom_s\, x$, if $Y(s) \leq v(s)$ and $Y_{is_i} > X_{is_i}$ for all $i \in car(s)=\{i \in N \mid s_i>0\}$. Further, we say that the imputation y dominates the imputation x if there exists $s \in \mathcal{M}^N \setminus \{0\}$ such that $y\, dom_s\, x$.

The *dominance core* $DC(v)$ of $v \in MC^{N,m}$ consists of all $x \in I(v)$ for which there exists no $y \in I(v)$ such that y dominates x, that is

$$DC(v)=\{x \in I(v) \mid \nexists y \in I(v) \text{ s.t. } y\, dom\, x\}.$$

A set $A \subset I(v)$ is a *stable set* if it is internally stable, that is $A \cap D(A) \neq \emptyset$, and it is externally stable, that is $I(v) \setminus A \subset D(A)$. Here, $D(A)=\{x \in I(v) \mid \exists a \in A \text{ s.t. } a\, dom\, x\}$.

Relations among the core, the dominance core and stable sets in the traditional cooperative game model still hold in the multi-choice model. In particular, the core of v is a subset of the dominance core of v; every stable set contains the dominance core as a subset; if the dominance core of v is a stable set, then there are no other stable sets. For additional results on cores and stable sets for multi-choice games the reader is referred to Part III in Branzei, Dimitrov and Tijs [1].

The *equal division core* $EDC(v)$ of $v \in MC^{N,m}$ is introduced in Branzei, Llorca, Sánchez-Soriano and Tijs [3] based on the (per-unit level) average worth, $\alpha(s,v)=v(s)/\sum_{i\in N} s_i$, of a multi-choice coalition $s \in \mathcal{M}^N \setminus \{0\}$ for the game v, that is

$$\begin{aligned}EDC(v)=\{x: M \to \mathbb{R} \mid X(m)=v(m);\\ \nexists s \in \mathcal{M}^N \setminus \{0\} \text{ s.t. } \alpha(s,v)>x_{ij}\\ \text{for all } i \in car(s),\ j \in M_i^+\}.\end{aligned}$$

It holds $C(v) \subset \mathcal{PC}(v) \subset EDC(v)$ for each $v \in MC^{N,m}$.

Another set-valued solution concept on $MC^{N,m}$ is the *equal split-off set* introduced in Branzei, Dimitrov and Tijs [2] as a straightforward generalization of the equal split-off set on G^N.

The multi-choice version of the *Weber set* $W(v)$ of $v \in G^N$ is defined as the convex hull of the (level) marginal vectors $w^{\sigma,v}$ corresponding to admissible orderings σ of players in $v \in MC^{N,m}$, i.e. orderings which take into account the fact that each player can reach a higher level of participation only via one-unit level increases starting from level 0. Thus, each admissible ordering σ of players generates a path from the participation profile $(0, ..., 0)$ to $(m_1, ..., m_n)$, along which the differences in worth for each one-unit level increase become $w_{ij}^{\sigma,v}$ for $i \in N$ and $j \in M_i^+$. We notice that marginal vectors $w^{\sigma,v}$ are not necessarily imputations. Given $v \in MC^{N,m}$ and $x \in C(v)$ it is proved by Nouweland et al. [16] that there is a $y \in W(v)$ that is weakly smaller than x. Thus, the relation between the core and the Weber set in the multi-choice game theory is different from that existing in the classical cooperative game theory, where the Weber set is a core catcher.

On the class of multi-choice games several solution concepts, which we call here *Shapley-like values*, are inspired basically by the *Shapley value* (cf. Shapley [19]). We briefly present here the most important ones.

The *Shapley value* Φ, a natural extension of the Shapley value on G^N, is introduced by Nouweland et al. [16] as the average of (level) marginal vectors, and axiomatically characterized by additivity, the carrier property and the hierarchical strength property. This value is further studied in Calvo and Santos [6] where the focus is on players' total payoffs instead of (level) payoff vectors. It is shown that this value corresponds to the discrete Aumann-Shapley method proposed in Moulin [14].

In Hsiao and Raghavan [11] the *Shapley value* Ψ^w is introduced, where w is a weight vector corresponding to players' levels under the assumptions of equal number of levels for all players and increasing ordering of weights with respect to levels. The Shapley values Ψ^w extend ideas of weighted Shapley values (cf. Kalai and Samet [12]). An axiomatic characterization of Ψ^w is provided using additivity, the carrier property, the minimal effort property and the weight property.

The *Shapley value* Θ is introduced by Derks and Peters [7]. In Klijn, Slikker and Zarzuelo [13] it is proved that Θ can be seen as the (level) payoff vector of average marginal contributions of the elements in $\mathcal{M}^N \setminus \{0\}$. The Shapley value Θ is axiomatically characterized in Nouweland [15] in the spirit of Young [22]; other axiomatic characterizations of it can be found in Klijn, Slikker and Zarzuelo [13].

The *Shapley value* ε, called the *egalitarian multi-choice solution*, is introduced by Peters and Zank [17] and axiomatically characterized by the properties of efficiency, zero-contribution, additivity and level-symmetry.

We also mention here the *multi-choice Shapley value* introduced by Grabisch and Lange [8].

More about the foregoing solution concepts is known on special classes of multi-choice games.

2. Solution concepts for convex multi-choice games

Convex multi-choice games are introduced in Nouweland et al. [16] as games whose characteristic function is supermodular. Formally, a game $v \in MC^{N,m}$ is convex if $v(s \wedge t) + v(s \vee t) \geq v(s) + v(t)$ for all $s, t \in \mathcal{M}^N$, where $(s \wedge t)_i = \min\{s_i, t_i\}$ and $(s \vee t)_i = \max\{s_i, t_i\}$ for all $i \in N$. It is shown that the core of a convex multi-choice game is the unique stable set of the game, and that a multi-choice game v is convex if and only if its Weber set equals the convex hull of the minimal core of the game, i.e. $W(v) = co(C_{\min}(v))$ holds. Consequently, the Shapley value $\Phi(v)$ of v belongs to the core $C(v)$ of v, in case v is convex. In Grabisch and Xie [9] notions related to the core and the Weber set for multi-choice games are defined in such a way that the equality between the core of a convex multi-choice game and the Weber set of that game still holds true.

Convexity of a multi-choice game proved to be a sufficient condition for the existence of monotonic allocation schemes (cf. Sprumont [20]) in a multi-choice setting. Such schemes, called (level-increase) monotonic allocation schemes (limas), are introduced and studied in Branzei, Tijs and Zarzuelo [5]. Let $v \in MC^{N,m}$ be a convex game.

A scheme $a = [a_{ij}^t]_{i \in N,\ j \in \{1,...,t_i\}}^{t \in \mathcal{M}^N \setminus \{0\}}$ is called a (*level-increase*) *monotonic allocation scheme* (*limas*) for v if it satisfies a stability condition, i.e. $a^t \in C(v_t)$ for each subgame v_t of v with $t \in \mathcal{M}^N \setminus \{0\}$, and a (level) monotonicity condition, i.e. $a_{ij}^s \leq a_{ij}^t$ for all $s, t \in \mathcal{M}^N \setminus \{0\}$ with $s \leq t$, each $i \in car(s)$, and each $j \in \{1, ..., s_i\}$. The subgame of $v \in MC^{N,m}$ with respect to $t \in \mathcal{M}^N \setminus \{0\}$ is defined by $v_t(s) := v(s)$ for each $s \in \mathcal{M}^N \setminus \{0\}$ such that $s \leq t$. We denote by $\mathcal{M}_t^N$ the subset of $\mathcal{M}^N \setminus \{0\}$ consisting of multi-choice coalitions $s \leq t$ and by M_i^t the set $\{1, ..., t_i\}$.

In particular, the total Shapley value (cf. Nouweland et al. [16]) of a convex multi-choice game, which is the scheme $[\Phi_{ij}(v_t)]_{i \in N,\ j \in \{1,...,t_i\}}^{t \in \mathcal{M}^N \setminus \{0\}}$ with the Shapley value of the multi-choice subgame

t in each row t, is a (level-increase) monotonic allocation scheme for v. It turns out that each element of the Weber set of a convex multi-choice game is extendable to a limas, that is there exists a limas $[a_{ij}^t]_{i\in N,\ j\in\{1,\ldots,t_i\}}^{t\in\mathcal{M}^N\setminus\{0\}}$ such that $a_{ij}^m = x_{ij}$ for each $i\in N$ and $j\in M_i^+$.

The *constrained egalitarian solution* is introduced on the class of convex multi-choice games in Branzei, Llorca, Sánchez-Soriano and Tijs [3] by using an adjusted version of the Dutta-Ray algorithm for traditional convex games based on the (per one-unit level-increase) average worth of a multi-choice coalition $s\in\mathcal{M}^N\setminus\{0\}$. Here, a key role is played by a proposition showing that there exists a unique multi-choice coalition with the largest aggregate number of levels of players among all coalitions with the highest (per one-unit level-increase) average worth. Then, a sequence of marginal games, each of which is a convex multi-choice game, is considered, that corresponds to the unique sequence of multi-choice coalitions in line with the above mentioned proposition. The marginal game of $v\in MC^{N,m}$ based on $u\in\mathcal{M}^N\setminus\{0\}$ is defined by $v^{-u}(s):=v(s+u)-v(u)$ for each $s\in\mathcal{M}^N\setminus\{0\}$ such that $s\le m-u$, that is for each $s\in\mathcal{M}_{m-u}^N$. Now, we formulate the Dutta-Ray algorithm for convex multi-choice games.

Step 1: Consider $m^1:=m$, $v_1:=v$. Select the unique element in arg $\max_{s\in\mathcal{M}_{m^1}^N\setminus\{0\}}\alpha(s,v_1)$ with the maximal aggregate number of levels, say s^1. Define $d_{ij}:=\alpha(s^1,v_1)$ for each $i\in car(s^1)$ and $j\in M_i^{s^1}$. If $s^1=m$, then stop; otherwise, go on.

Step p: Suppose that $s^1,s^2,\ldots,s^{p-1}$ have been defined recursively and $s^1+s^2+\ldots+s^{p-1}\neq m$. Define a new multi-choice game with player set N and maximal participation profile $m^p:=m-\sum_{i=1}^{p-1}m^i$. For each multi-choice coalition $s\in\mathcal{M}_{m^p}^N$, define $v_p(s):=v_{p-1}(s+s^{p-1})-v_{p-1}(s^{p-1})$. The game $v_p\in MC^{N,m^p}$ is convex. Denote by s^p the (unique) largest element in arg $\max_{s\in\mathcal{M}_{m^p}^N\setminus\{0\}}\alpha(s,v_p)$ and define $d_{ij}:=\alpha(s^p,v_p)$ for all $i\in car(s^p)$ and $j\in\left\{\sum_{k=1}^{p-1}s_i^k+1,\ldots,\sum_{k=1}^{p}s_i^k\right\}$.

In a finite number of steps, say P, where $P\le|M|$, $M=\{(i,j)\mid i\in N,\ j\in M_i\}$, and $|M|$ is the cardinality of the set M, the algorithm will end, and the constructed (level) payoff vector $(d_{ij})_{(i,j)\in M^+}$ is called the (*Dutta-Ray*) *constrained egalitarian solution* $d(v)$ of the convex multi-choice game v. It is proved that the constrained egalitarian solution for convex multi-choice games has similar properties as the constrained egalitarian solution for traditional convex games. Specifically, the constrained egalitarian allocation is a Lorenz undominated element of the precore, and also belongs to the equal division core of the game. We notice that the role of the core for a convex game in G^N is played now by the precore of a convex multi-choice game. However, it is still an open question whether the constrained egalitarian solution of a convex multi-choice game possesses a population monotonicity property regarding players' levels of participation. It turns out that for each convex multi-choice game v the equal split-off set $ESOS(v)$ consists of a unique equal split-off allocation which equals the constrained egalitarian solution of that game, i.e. $ESOS(v)=\{d(v)\}$ for each convex game $v\in MC^{N,m}$.

3. Solution concepts for multi-choice total clan games

Multi-choice clan games are introduced in Branzei, Llorca, Sánchez-Soriano and Tijs [4] to extend the model of traditional clan games (cf. Potters, Poos, Tijs and Muto [18]). In a multi-choice clan game the set N of players consists of two disjoint groups: a fixed (powerful) clan C with 'yes-or-no' choices, and a group of (nonpowerful) non-clan members having more possibilities for being active. Multi-choice clan games are defined using the veto power of clan members, the monotonicity property of the characteristic function, and a (level) union property regarding non-clan members' participation in multi-choice coalitions containing at least all clan members at participation level 1. We denote by $\mathcal{M}^{N,C}$ the set of multi-choice coalitions with player set N and fixed clan C, and by $\mathcal{M}^{N,1_C}$ the set of all multi-choice coalitions containing at least all clan members at participation level 1. For each $s\in\mathcal{M}^{N,C}$ we denote its restrictions to $N\setminus C$ and C, by $s_{N\setminus C}$ and s_C, respectively. Clearly, the maximal participation profile of players in a multi-choice clan game with fixed player set N and fixed clan C is of the form $m=(m_{N\setminus C},1_C)$. Formally, a game $<N,(m_{N\setminus C},1_C),v>$ is a multi-choice clan game if v satisfies:

(i) Clan property: $v(s) = 0$ if $s_C \neq 1_C$;

(ii) Monotonicity property: $v(s) \leq v(t)$ for all $s, t \in \mathcal{M}^{N,C}$ with $s \leq t$;

(iii) (Level) Union property: For each $s \in \mathcal{M}^{N,1_C}$, $v(m) - v(s) \geq \sum_{i \in N \setminus C} (v(m) - v(m_{-i}, s_i))$, where (m_{-i}, s_i) is the multi-choice coalition where all players $j \in N \setminus C$, $j \neq i$, participate at their maximal level m_j, whereas non-clan member i participates at his level s_i in s.

We denote the set of multi-choice clan games with player set N, fixed clan C and maximal participation profile $m = (m_{N \setminus C}, 1_C)$ by $MC_C^{N,m}$.

The core of a multi-choice game $v \in MC_C^{N,m}$ is explicitly described as

$$C(v) = \{x : M \to \mathbb{R}_+ \mid X(m) = v(m); \sum_{k=j}^{m_i} x_{ik} \leq v(m) - v(m_{-i}, j-1), \text{ for all } i \in N \setminus C,\, j \in M_i^+\}.$$

A multi-choice total clan game is a clan game whose all subgames are also clan games. The subgame of $v \in MC_C^{N,m}$ with respect to $t \in \mathcal{M}^{N,1_C}$ is defined by $v_t(s) := v(s)$ for each $s \in \mathcal{M}_t^{N,1_C}$, where $\mathcal{M}_t^{N,1_C}$ stands for the subset of $\mathcal{M}^{N,1_C}$ with $s_{N \setminus C} \leq t_{N \setminus C}$. The structure of the core of a multi-choice total clan game and that of the core of its subgames play an important role for the existence of bi-monotonic allocation schemes for such games. A (level) total concavity property of multi-choice total clan games also plays a role for the existence of bi-monotonic allocation schemes for such games:

For $s, t \in \mathcal{M}^{N,1_C}$ with $s \leq t$ and for each $i \in car(s_{N \setminus C})$ such that $s_i = t_i$ it holds

$$v(t) - v(t - e^i) \leq v(s) - v(s - e^i).$$

This property reflects the fact that the same one-unit level decrease of a non-clan member in coalitions containing at least all clan members at participation level 1 and where that non-clan member has the same participation level, could be more beneficial in smaller such coalitions than in larger ones. It turns out that, for multi-choice games possessing both the clan property and the monotonicity property, the total concavity property is equivalent with the total (level) union property:

For all $s, t \in \mathcal{M}^{N,1_C}$ with $s \leq t$ it holds:

$$v(t) - v(s) \geq \sum_{i \in car(t_{N \setminus C})} (v(t) - v(t_{-i}, s_i)).$$

A scheme $b = [b_{ij}^t]_{i \in N,\ j \in \{1,\ldots,t_i\}}^{t \in \mathcal{M}^{N,1_C}}$ is called a *bi-(level-increase) monotonic allocation scheme* (*bi-limas*) if it satisfies a stability condition, i.e. $b^t \in C(v_t)$ for each subgame v_t of v with $t \in \mathcal{M}^{N,1_C}$, and a (level) bi-monotonicity property regarding the two types of players, i.e. for all $s, t \in \mathcal{M}^{N,1_C}$ with $s \leq t$ it holds: (i)$b_{i1}^s \leq b_{i1}^t$ for each $i \in C$, and (ii) $b_{ij}^s \geq b_{ij}^t$ for each $i \in car(s_{N \setminus C})$ and each $j \in \{1, ..., s_i\}$.

This kind of bi-monotonic allocation schemes are introduced in Branzei, Llorca, Sánchez-Soriano and Tijs [4] and studied by means of suitably defined compensation-sharing rules $\psi^{\alpha,\beta} : MC_C^{N,m} \to \mathbb{R}^{|M|}$, where $\alpha \in [0,1]^{N \setminus C}$ and $\beta \in \Delta(C)$, with $\Delta(C)$ being the unit simplex whose coordinates correspond to clan members.

The i-th coordinate α_i of the compensation vector α indicates the share, to be given to level 1 of non-clan member i, of i's contribution to the grand coalition m, whereas the i-th coordinate β_i of the sharing vector β determines the share of the remainder for the clan given to clan member i. It turns out that for a subclass of multi-choice total clan games compensation-sharing rules $\psi^{\alpha,\beta}$ with $\alpha \in [0,1]^{N \setminus C}$ and $\beta \in N(C)$ generate bi-(level-increase) monotonic allocation schemes. Furthermore, some elements x in the core of each multi-choice game in that subclass of total clan games are extendable to a bi-limas, that is there exists a bi-limas $[b_{ij}^t]_{i \in N,\ j \in \{1,\ldots,t_i\}}^{t \in \mathcal{M}^{N,1_C}}$ such that $b_{ij}^m = x_{ij}$ for each $i \in N$, $j \in M_i^+$. Clearly, when $s_{N \setminus C} = 1_{N \setminus C}$ a bi-limas coincides with a bi-mas (cf. Voorneveld, Tijs and Grahn [21]), and we obtain as a particular case that each core element of a total clan game in G^N is extendable to a bi-mas.

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